

The Intergals of Motion for the Deformed W -Algebra $W_{q,t}(\widehat{sl}_N)$ II:

Proof of the commutation relations

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Dedicated to Professor Tetsuji Miwa on the occasion on the 60th birthday

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Abstract

We explicitly construct two classes of infinitely many commutative operators in terms of the deformed W -algebra $W_{q,t}(\widehat{sl}_N)$, and give proofs of the commutation relations of these operators. We call one of them local integrals of motion and the other nonlocal one, since they can be regarded as elliptic deformations of local and nonlocal integrals of motion for the W_N algebra [1, 2].

1 Introduction

This is a continuation of the papers [13, 14], hereafter referred to as Part 1 [13] and Part 2 [14]. In Part 1 we constructed two classes of infinitely many commutative operators, in terms of the deformed Virasoro algebra. In Part 2 we announced conjecturous formulae of two classes of infinitely many commutative operators, in terms of the deformed W algebra $W_{q,t}(\widehat{sl}_N)$, which is the higher-rank generalization of Part 1 [13]. We call one of them local integrals of motion and the other nonlocal one, since they can be regarded as elliptic deformations of local and nonlocal integrals of motion for the W_N algebra [1, 2]. In this paper we give proofs of the commutation relations for the integrals of motion for the deformed W algebra $W_{q,t}(\widehat{sl}_N)$.

Let us recall some facts about soliton equation and its quantization. B. Feigin and E. Frenkel [3] considered the so-called local integrals of motion $I^{(cl)}$ for the Toda field theory associated with the root system of finite and affine type $\{I^{(cl)}, H^{(cl)}\}_{P.B.} = 0$, where $H^{(cl)} = \frac{1}{2} \int (e^{\phi(t)} + e^{-\phi(t)}) dt$ is the Hamiltonian of the Toda field theory. They showed the existence of infinitely many commutative integrals of motion by a cohomological argument, and showed that they can be regarded as the conservation laws for the generalized KdV equation. In [3] they constructed the quantum deformation of the local integrals of motion, too. In other words they showed the existence of quantum deformation of the conservation laws of the generalized KdV equation. After quantization Gel'fand-Dickij bracket $\{, \}_{P.B.}$ for the second Hamiltonian structure of the generalized KdV, gives rise to the W_N algebra. V. Bazhanov et.al [1, 2] constructed field theoretical analogue of the commuting transfer matrix $\mathbf{T}(z)$, acting on the irreducible highest weight module of the Virasoro algebra and the W_3 algebra. They constructed this commuting transfer matrix $\mathbf{T}(z)$ as the trace of the monodromy matrix associated with the quantum affine symmetry $U_q(\widehat{sl}_2)$ and $U_q(\widehat{sl}_3)$, and showed that the commutativity relation $[\mathbf{T}(z), \mathbf{T}(w)] = 0$ is a direct consequence of the Yang-Baxter relation. The coefficients of the asymptotic expansion of the operator $\log \mathbf{T}(z)$ at $z \rightarrow \infty$, produce the local integrals of motion for the Virasoro algebra and the W_3 algebra, which reproduce the conservation laws of the generalized KdV equation in the classical limit $c_{CFT} \rightarrow \infty$. They call the coefficients of the Taylor expansion of the operator $\mathbf{T}(z)$ at $z = 0$, the nonlocal integrals of motion for the Virasoro algebra and the W_3 algebra. They have explicit integral representation of

the nonlocal integrals in terms of the screening currents.

The purpose of this paper is to construct the elliptic version of the integrals of motion given by Bazhanov et.al [1, 2]. Bazhanov et.al's construction is based on the free field realization of the Borel subalgebra \mathcal{B}_\pm of $U_q(\widehat{sl_2})$ and $U_q(\widehat{sl_3})$. By using this realization they construct the monodromy matrix as the image of the universal R-matrix $\bar{R} \in \mathcal{B}_+ \otimes \mathcal{B}_-$, and make the transfer matrix $\mathbf{T}(z)$ as the trace of the monodromy matrix. The universal R-matrix \bar{R} of the elliptic quantum group does not exist in $\mathcal{B}_+ \otimes \mathcal{B}_-$. Hence it is impossible to construct the elliptic deformation of the transfer matrix $\mathbf{T}(z)$ as the same manner as [1]. Our method of construction should be completely different from those of [1, 2]. Instead of considering the transfer matrix $\mathbf{T}(z)$, we directly give the integral representations of the integrals of motion $\mathcal{I}_n, \mathcal{G}_n$ for the deformed Virasoro algebra. The commutativity of our integrals of motion are not understood as a direct consequence of the Yang-Baxter equation. They are understood as a consequence of the commutative subalgebra of the Feigin-Odesskii algebra [8].

The organization of this paper is as follows. In Section 2, we review the deformed W algebra, including free field realization, screening currents [4, 6]. In Section 3, we give integral representations for the local integrals of motion \mathcal{I}_n , and show the commutation relations :

$$[\mathcal{I}_m, \mathcal{I}_n] = [\mathcal{I}_m^*, \mathcal{I}_n^*] = 0.$$

Very precisely, in Part 2 [14], we only give the Laurent series representation of the local integrals motion, which is useful for proofs of the commutation relation and Dynkin-automorphism invariance. In this section we show the integral representations and the Laurent series representation give the same local integrals of motion. In Section 4, we give explicit formulae for the nonlocal integrals of motion \mathcal{G}_n , and show the commutation relations :

$$[\mathcal{G}_m, \mathcal{G}_n] = [\mathcal{G}_m^*, \mathcal{G}_n^*] = [\mathcal{G}_m, \mathcal{G}_n^*] = 0,$$

$$[\mathcal{I}_m, \mathcal{G}_n] = [\mathcal{I}_m^*, \mathcal{G}_n] = [\mathcal{I}_m, \mathcal{G}_n^*] = [\mathcal{I}_m^*, \mathcal{G}_n^*] = 0.$$

We show the commutation relation $[\mathcal{I}_m, \mathcal{G}_n] = 0$ using Dynkin-automorphism invariance $\eta(\mathcal{I}_n) = \mathcal{I}_n$ and $\eta(\mathcal{G}_n) = \mathcal{G}_n$, which will be shown in the next section. In Section 5, we give proofs of Dynkin-automorphism invariance :

$$\eta(\mathcal{I}_n) = \mathcal{I}_n, \quad \eta(\mathcal{I}_n^*) = \mathcal{I}_n^*, \quad \eta(\mathcal{G}_n) = \mathcal{G}_n, \quad \eta(\mathcal{G}_n^*) = \mathcal{G}_n^*.$$

In Appendix we summarize the normal ordering of the basic operators. We would like to point out a differnt point between the case of the deformed Virasoro $Vir_{q,t} = W_{q,t}(\widehat{sl_2})$ and its higher-rank generalization : the deformed $W_{q,t}(\widehat{sl_N})$, ($N \geq 3$). Basically situations of $W_{q,t}(\widehat{sl_N})$, ($N \geq 3$) are more complicated than those of $Vir_{q,t} = W_{q,t}(\widehat{sl_2})$. However, one thing of $W_{q,t}(\widehat{sl_N})$, ($N \geq 3$) is simpler than those of $Vir_{q,t} = W_{q,t}(\widehat{sl_2})$. In the case of $Vir_{q,t} = W_{q,t}(\widehat{sl_2})$, the integrals of motions $\mathcal{I}_n, \mathcal{G}_n$ have singularity at $s = N = 2$. Hence we considered the renormalized limits for the integral of motions $\mathcal{I}_n, \mathcal{G}_n$ in the lase section of the paper [13]. In the case of $W_{q,t}(\widehat{sl_N})$, ($N \geq 3$), the integrals of motions $\mathcal{I}_n, \mathcal{G}_n$ do not have singularity at $s = N \geq 3$.

At the end of Introduction, we would like to mention about two important degenerating limits of the deformed W algebra. One is the CFT-limit[1, 2] and the other is the classical limit[11]. In the CFT-limit V.Bazhanov et.al. [1, 2] constructed infinitely many integrals of motion for the Virasoro algebra, as we mentioned above. In the classical limit, the deformed Virasoro algebra degenerates to the Poisson-Virasoro algebra introduced by E.Frenkel and N.Reshetikhin [11].

2 The Deformed W -Algebra $W_{q,t}(\widehat{sl_N})$

In this section we review the deformed W -algebra and its screening currents. We prepare the notations to be used in this paper. Throughout this paper, we fix generic three parameters $0 < x < 1$, $r \in \mathbb{C}$ and $s \in \mathbb{C}$. Let us set $z = x^{2u}$. Let us set $r^* = r - 1$. The symbol $[u]_r$ for $\text{Re}(r) > 0$ stands for the Jacobi theta function

$$[u]_r = x^{\frac{u^2}{r}-u} \frac{\Theta_{x^{2r}}(x^{2u})}{(x^{2r}; x^{2r})_3}, \quad \Theta_q(z) = (z; q)_\infty (qz^{-1}; q)_\infty (q; q)_\infty, \quad (2.1)$$

where we have used the standard notation

$$(z; q)_\infty = \prod_{j=0}^{\infty} (1 - q^j z). \quad (2.2)$$

We set the parametrizations τ, τ^*

$$x = e^{-\pi\sqrt{-1}/r\tau} = e^{-\pi\sqrt{-1}/r^*\tau^*}. \quad (2.3)$$

The theta function $[u]_r$ enjoys the quasi-periodicity property

$$[u+r]_r = -[u]_r, \quad [u+r\tau]_r = -e^{-\pi\sqrt{-1}\tau - \frac{2\pi\sqrt{-1}}{r}u} [u]_r. \quad (2.4)$$

The symbol $[a]$ stands for

$$[a] = \frac{x^a - x^{-a}}{x - x^{-1}}. \quad (2.5)$$

2.1 Free Field Realization

Let $\epsilon_i (1 \leq i \leq N)$ be an orthonormal basis in \mathbb{R}^N relative to the standard basis in \mathbb{R}^N relative to the standard inner product $(,)$. Let us set $\bar{\epsilon}_i = \epsilon_i - \epsilon$, $\epsilon = \frac{1}{N} \sum_{j=1}^N \epsilon_j$. We identify $\epsilon_{N+1} = \epsilon_1$. Let $P = \sum_{i=1}^N \mathbb{Z} \bar{\epsilon}_i$ the weight lattice. Let us set $\alpha_i = \bar{\epsilon}_i - \bar{\epsilon}_{i+1} \in P$.

Let β_m^j be the oscillators $(1 \leq j \leq N, m \in \mathbb{Z} - \{0\})$ with the commutation relations

$$[\beta_m^i, \beta_n^j] = \begin{cases} m \frac{[(r-1)m]}{[rm]} \frac{[(s-1)m]}{[sm]} \delta_{n+m,0} & (1 \leq i = j \leq N) \\ -m \frac{[(r-1)m]}{[rm]} \frac{[m]}{[sm]} x^{sm} \operatorname{sgn}(i-j) \delta_{n+m,0} & (1 \leq i \neq j \leq N) \end{cases} \quad (2.6)$$

We also introduce the zero mode operator P_λ , $(\lambda \in P)$. They are \mathbb{Z} -linear in P and satisfy

$$[iP_\lambda, Q_\mu] = (\lambda, \mu), \quad (\lambda, \mu \in P). \quad (2.7)$$

Let us intrduce the bosonic Fock space $\mathcal{F}_{l,k}(l, k \in P)$ generated by $\beta_{-m}^j (m > 0)$ over the vacuum vector $|l, k\rangle$:

$$\mathcal{F}_{l,k} = \mathbb{C}[\{\beta_{-1}^j, \beta_{-2}^j, \dots\}_{1 \leq j \leq N}] |l, k\rangle, \quad (2.8)$$

where

$$\beta_m^j |l, k\rangle = 0, (m > 0), \quad (2.9)$$

$$P_\alpha |l, k\rangle = \left(\alpha, \sqrt{\frac{r}{r-1}} l - \sqrt{\frac{r-1}{r}} k \right) |l, k\rangle, \quad (2.10)$$

$$|l, k\rangle = e^{i\sqrt{\frac{r}{r-1}} Q_l - i\sqrt{\frac{r-1}{r}} Q_k} |0, 0\rangle. \quad (2.11)$$

Let us set the Dynkin-diagram automorphism η by

$$\eta(\beta_m^1) = x^{-\frac{2s}{N}m} \beta_m^2, \dots, \eta(\beta_m^{N-1}) = x^{-\frac{2s}{N}m} \beta_m^N, \quad \eta(\beta_m^N) = x^{\frac{2s}{N}(N-1)m} \beta_m^1, \quad (2.12)$$

and $\eta(\epsilon_i) = \epsilon_{i+1}$, $(1 \leq i \leq N)$.

2.2 The Deformed W -Algebra

In this section we give short review of the deformed W -algebra $W_{q,t}(\widehat{sl_N})$ [5, 6, 7].

Definition 2.1 We set the fundamental operator $\Lambda_j(z)$, $(1 \leq j \leq N)$ by

$$\Lambda_j(z) = x^{-2\sqrt{r(r-1)}P_{\bar{\epsilon}_j}} : \exp \left(\sum_{m \neq 0} \frac{x^{rm} - x^{-rm}}{m} \beta_m^j z^{-m} \right) : \quad (1 \leq j \leq N). \quad (2.13)$$

Definition 2.2 Let us set the operator $T_j(z)$, $(1 \leq j \leq N)$ by

$$T_j(z) = \sum_{1 \leq s_1 < s_2 < \dots < s_j \leq N} : \Lambda_{s_1}(x^{-j+1}z) \Lambda_{s_2}(x^{-j+3}z) \dots \Lambda_{s_j}(x^{j-1}z) : . \quad (2.14)$$

Proposition 2.1 The actions of η on the fundamental operators $\Lambda_j(z)$, $(1 \leq j \leq N)$ are given by

$$\eta(\Lambda_j(z)) = \Lambda_{j+1}(x^{\frac{2s}{N}}z), \quad (1 \leq j \leq N-1), \quad \eta(\Lambda_N(z)) = \Lambda_1(x^{\frac{2s}{N}-2s}z). \quad (2.15)$$

Proposition 2.2 The operators $T_j(z)$, $(1 \leq j \leq N)$ satisfy the following relations.

$$\begin{aligned} & f_{i,j}(z_2/z_1)T_i(z_1)T_j(z_2) - f_{j,i}(z_1/z_2)T_j(z_2)T_i(z_1) \\ &= c \sum_{k=1}^i \prod_{l=1}^{k-1} \Delta(x^{2l+1}) \times \left(\delta \left(\frac{x^{j-i+2k}z_2}{z_1} \right) f_{i-k,j+k}(x^{-j+i})T_{i-k}(x^{-k}z_1)T_{j+k}(x^kz_2) \right. \\ & \quad \left. - \delta \left(\frac{x^{-j+i-2k}z_2}{z_1} \right) f_{i-k,j+k}(x^{j-i})T_{i-k}(x^kz_1)T_{j+k}(x^{-k}z_2) \right), \quad (1 \leq i \leq j \leq N) \end{aligned} \quad (2.16)$$

where we have used the delta-function $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$. Here we set the constnt c and the auxiliary function $\Delta(z)$ by

$$c = -\frac{(1-x^{2r})(1-x^{-2r+2})}{(1-x^2)}, \quad \Delta(z) = \frac{(1-x^{2r-1}z)(1-x^{1-2r}z)}{(1-xz)(1-x^{-1}z)}. \quad (2.17)$$

Here we set the structure functions,

$$f_{i,j}(z) = \exp \left(\sum_{m=1}^{\infty} \frac{1}{m} (1-x^{2rm})(1-x^{-2(r-1)m}) \frac{(1-x^{2m \min(i,j)})(1-x^{2m(s-\max(i,j))})}{(1-x^{2m})(1-x^{2sm})} x^{|i-j|m} z^m \right). \quad (2.18)$$

Above proposition is one parameter “ s ” generalization of [7]. The proof is given by the same manner.

Example For $N = 2$ the operators $T_1(z), T_2(z)$ satisfy

$$\begin{aligned} & f_{1,1}(z_2/z_1)T_1(z_1)T_1(z_2) - f_{1,1}(z_1/z_2)T_1(z_2)T_1(z_1) \\ &= c(\delta(x^2 z_2/z_1)T_2(xz_2) - \delta(x^2 z_1/z_2)T_2(x^{-1}z_2)), \end{aligned} \quad (2.19)$$

$$f_{1,2}(z_2/z_1)T_1(z_1)T_2(z_2) = f_{2,1}(z_1/z_2)T_2(z_2)T_1(z_1), \quad (2.20)$$

$$f_{2,2}(z_2/z_1)T_2(z_1)T_2(z_2) = f_{2,2}(z_1/z_2)T_2(z_2)T_2(z_1). \quad (2.21)$$

Example For $N = 3$ the operators $T_1(z), T_2(z), T_3(z)$ satisfy

$$\begin{aligned} & f_{1,1}(z_2/z_1)T_1(z_1)T_1(z_2) - f_{1,1}(z_1/z_2)T_1(z_2)T_1(z_1) \\ &= c(\delta(x^2 z_2/z_1)T_2(xz_2) - \delta(x^2 z_1/z_2)T_2(x^{-1}z_2)), \end{aligned} \quad (2.22)$$

$$\begin{aligned} & f_{1,2}(z_2/z_1)T_1(z_1)T_2(z_2) - f_{2,1}(z_1/z_2)T_2(z_2)T_1(z_1) \\ &= c(\delta(x^3 z_2/z_1)T_3(xz_2) - \delta(x^3 z_1/z_2)T_3(x^{-1}z_2)), \end{aligned} \quad (2.23)$$

$$\begin{aligned} & f_{2,2}(z_2/z_1)T_2(z_1)T_2(z_2) - f_{2,2}(z_1/z_2)T_2(z_2)T_2(z_1) \\ &= cf_{1,3}(1)(\delta(x^2 z_2/z_1)T_1(xz_2)T_3(xz_2) - \delta(x^2 z_1/z_2)T_1(x^{-1}z_2)T_3(x^{-1}z_2)), \end{aligned} \quad (2.24)$$

$$f_{1,3}(z_2/z_1)T_1(z_1)T_3(z_2) = f_{3,1}(z_1/z_2)T_3(z_2)T_1(z_1), \quad (2.25)$$

$$f_{2,3}(z_2/z_1)T_2(z_1)T_3(z_2) = f_{3,2}(z_1/z_2)T_3(z_2)T_2(z_1), \quad (2.26)$$

$$f_{3,3}(z_2/z_1)T_3(z_1)T_3(z_2) = f_{3,3}(z_1/z_2)T_3(z_2)T_3(z_1). \quad (2.27)$$

Example For $N = 4$ the operators $T_1(z), T_2(z), T_3(z), T_4(z)$ satisfy

$$\begin{aligned} & f_{1,1}(z_2/z_1)T_1(z_1)T_1(z_2) - f_{1,1}(z_1/z_2)T_1(z_2)T_1(z_1) \\ &= c(\delta(x^2 z_2/z_1)T_2(xz_2) - \delta(x^2 z_1/z_2)T_2(x^{-1}z_2)), \end{aligned} \quad (2.28)$$

$$\begin{aligned} & f_{1,2}(z_2/z_1)T_1(z_1)T_2(z_2) - f_{2,1}(z_1/z_2)T_2(z_2)T_1(z_1) \\ &= c(\delta(x^3 z_2/z_1)T_3(xz_2) - \delta(x^3 z_1/z_2)T_3(x^{-1}z_2)), \end{aligned} \quad (2.29)$$

$$\begin{aligned} & f_{1,3}(z_2/z_1)T_1(z_1)T_3(z_2) - f_{3,1}(z_1/z_2)T_3(z_2)T_1(z_1) \\ &= c(\delta(x^4 z_2/z_1)T_4(xz_2) - \delta(x^4 z_1/z_2)T_4(x^{-1}z_2)), \end{aligned} \quad (2.30)$$

$$\begin{aligned} & f_{2,2}(z_2/z_1)T_2(z_1)T_2(z_2) - f_{2,2}(z_1/z_2)T_2(z_2)T_2(z_1) \\ &= cf_{1,3}(1)(\delta(x^2 z_2/z_1)T_1(x^{-1}z_1)T_3(xz_2) - \delta(x^2 z_1/z_2)T_1(xz_1)T_3(x^{-1}z_2)), \end{aligned} \quad (2.31)$$

$$\begin{aligned} & f_{2,3}(z_2/z_1)T_2(z_1)T_3(z_2) - f_{3,2}(z_1/z_2)T_3(z_2)T_2(z_1) \\ &= c(\delta(x^3 z_2/z_1)f_{1,4}(x^{-1})T_1(x^{-1}z_1)T_4(xz_2) - \delta(x^3 z_1/z_2)f_{1,4}(x)T_1(xz_1)T_4(x^{-1}z_2)), \end{aligned} \quad (2.32)$$

$$f_{1,4}(z_2/z_1)T_1(z_1)T_4(z_2) = f_{4,1}(z_1/z_2)T_4(z_2)T_1(z_1), \quad (2.33)$$

$$f_{2,4}(z_2/z_1)T_2(z_1)T_4(z_2) = f_{4,2}(z_1/z_2)T_4(z_2)T_2(z_1), \quad (2.34)$$

$$f_{3,4}(z_2/z_1)T_3(z_1)T_4(z_2) = f_{4,3}(z_1/z_2)T_4(z_2)T_3(z_1), \quad (2.35)$$

$$f_{4,4}(z_2/z_1)T_4(z_1)T_4(z_2) = f_{4,4}(z_1/z_2)T_4(z_2)T_4(z_1). \quad (2.36)$$

Definition 2.3 The deformed W -algebra is defined by the generators $\widehat{T}_m^{(j)}$, ($1 \leq j \leq N, m \in \mathbb{Z}$) with the defining relations (2.16). Here we should understand $\widehat{T}_m^{(j)}$ as the Fourier coefficients of the operators $\widehat{T}_j(z) = \sum_{m \in \mathbb{Z}} \widehat{T}_m^{(j)} z^{-m}$, ($1 \leq j \leq N$).

2.3 Screening Currents

In this section we introduce the screening currents $E_j(z)$ and $F_j(z)$.

Definition 2.4 We set the screening currents $F_j(z)$, ($1 \leq j \leq N$) by

$$\begin{aligned} F_j(z) &= e^{i\sqrt{\frac{r-1}{r}}Q_{\alpha_j}(x^{(\frac{2s}{N}-1)j}z)}\sqrt{\frac{r-1}{r}}P_{\alpha_j+\frac{r-1}{r}} \\ &\times : \exp\left(\sum_{m \neq 0} \frac{1}{m} B_m^j z^{-m}\right) :, \quad (1 \leq j \leq N-1) \end{aligned} \quad (2.37)$$

$$\begin{aligned} F_N(z) &= e^{i\sqrt{\frac{r-1}{r}}Q_{\alpha_N}(x^{2s-N}z)}\sqrt{\frac{r-1}{r}}P_{\epsilon_N+\frac{r-1}{2r}}z^{-\sqrt{\frac{r-1}{r}}P_{\epsilon_1+\frac{r-1}{2r}}} \\ &\times : \exp\left(\sum_{m \neq 0} \frac{1}{m} B_m^N z^{-m}\right) :, \end{aligned} \quad (2.38)$$

We set the screening currents $E_j(z)$, ($1 \leq j \leq N$) by

$$\begin{aligned} E_j(z) &= e^{-i\sqrt{\frac{r}{r-1}}Q_{\alpha_j}(x^{(\frac{2s}{N}-1)j}z)}^{-\sqrt{\frac{r}{r-1}}P_{\alpha_j+\frac{r}{r-1}}} \\ &\times : \exp\left(-\sum_{m \neq 0} \frac{1}{m} \frac{[rm]}{[(r-1)m]} B_m^j z^{-m}\right) :, \quad (1 \leq j \leq N-1) \end{aligned} \quad (2.39)$$

$$\begin{aligned} E_N(z) &= e^{-i\sqrt{\frac{r}{r-1}}Q_{\alpha_N}(x^{2s-N}z)}^{-\sqrt{\frac{r}{r-1}}P_{\epsilon_N+\frac{r}{2(r-1)}}}z^{\sqrt{\frac{r}{r-1}}P_{\epsilon_1+\frac{r}{2(r-1)}}} \\ &\times : \exp\left(-\sum_{m \neq 0} \frac{1}{m} \frac{[rm]}{[(r-1)m]} B_m^N z^{-m}\right) :. \end{aligned} \quad (2.40)$$

Here we have set

$$B_m^j = (\beta_m^j - \beta_m^{j+1})x^{-\frac{2s}{N}jm}, \quad (1 \leq j \leq N-1), \quad (2.41)$$

$$B_m^N = (x^{-2sm}\beta_m^N - \beta_m^1). \quad (2.42)$$

The screening currents $F_j(z), E_j(z)$ ($1 \leq j \leq N-1$) have already been studied in [9, 10]. We introduce new screening current $F_N(z), E_N(z)$, which can be regarded as “affinization” of screening currents $F_j(z), E_j(z)$ ($1 \leq j \leq N-1$). The following commutation relations are convenient for calculations.

$$[\beta_m^j, B_n^j] = m\delta_{m+n,0} \frac{[r^*m]}{[rm]} x^{(-1+\frac{2s}{N}j)m}, \quad (1 \leq j \leq N) \quad (2.43)$$

$$[\beta_m^{j+1}, B_n^j] = -m\delta_{m+n,0} \frac{[r^*m]}{[rm]} x^{(1+\frac{2s}{N}j)m}, \quad (1 \leq j \leq N-1) \quad (2.44)$$

$$[\beta_m^1, B_n^N] = -m\delta_{m+n,0} \frac{[r^*m]}{[rm]} x^m, \quad (2.45)$$

$$[B_m^j, B_n^j] = m\delta_{m+n,0} \frac{[r^*m]}{[rm]} \frac{[2m]}{[m]}, \quad (1 \leq j \leq N) \quad (2.46)$$

$$[B_m^j, B_n^{j+1}] = -m\delta_{m+n,0} \frac{[r^*m]}{[rm]} x^{(-1+\frac{2s}{N}j)m}, \quad (1 \leq j \leq N). \quad (2.47)$$

Here we read $B_m^{N+1} = B_m^1$. We summarize the commutation relations of the screening currents for $N \geq 3$.

Proposition 2.3 *The screening currents $F_j(z)$, ($1 \leq j \leq N; N \geq 3$) satisfy the following commutation relations for $\text{Re}(r) > 0$*

$$\frac{1}{[u_1 - u_2 - \frac{s}{N} + 1]_r} F_j(z_1) F_{j+1}(z_2) = \frac{1}{[u_2 - u_1 + \frac{s}{N}]_r} F_{j+1}(z_2) F_j(z_1), \quad (1 \leq j \leq N), \quad (2.48)$$

$$\frac{[u_1 - u_2]_r}{[u_1 - u_2 - 1]_r} F_j(z_1) F_j(z_2) = \frac{[u_2 - u_1]_r}{[u_2 - u_1 - 1]_r} F_j(z_2) F_j(z_1), \quad (1 \leq j \leq N), \quad (2.49)$$

$$F_i(z_1) F_j(z_2) = F_j(z_2) F_i(z_1), \quad (|i - j| \geq 2). \quad (2.50)$$

We read $F_{N+1}(z) = F_1(z)$. The screening currents $F_j(z)$, ($1 \leq j \leq N; N \geq 3$) satisfy the following commutation relations for $\text{Re}(r) < 0$.

$$\frac{1}{[u_1 - u_2 - \frac{s}{N}]_{-r}} F_j(z_1) F_{j+1}(z_2) = \frac{1}{[u_2 - u_1 - 1 + \frac{s}{N}]_{-r}} F_{j+1}(z_2) F_j(z_1), \quad (1 \leq j \leq N), \quad (2.51)$$

$$\frac{[u_1 - u_2]_{-r}}{[u_1 - u_2 + 1]_{-r}} F_j(z_1) F_j(z_2) = \frac{[u_2 - u_1]_{-r}}{[u_2 - u_1 + 1]_{-r}} F_j(z_2) F_j(z_1), \quad (1 \leq j \leq N), \quad (2.52)$$

$$F_i(z_1) F_j(z_2) = F_j(z_2) F_i(z_1), \quad (|i - j| \geq 2). \quad (2.53)$$

We read $F_{N+1}(z) = F_1(z)$.

The screening currents $E_j(z)$, $(1 \leq j \leq N; N \geq 3)$ satisfy the following commutation relations for $\text{Re}(r^*) > 0$

$$\frac{1}{[u_1 - u_2 - \frac{s}{N}]_{r^*}} E_j(z_1) E_{j+1}(z_2) = \frac{1}{[u_2 - u_1 + \frac{s}{N} - 1]_{r^*}} E_{j+1}(z_2) E_j(z_1), \quad (1 \leq j \leq N), \quad (2.54)$$

$$\frac{[u_1 - u_2]_{r^*}}{[u_1 - u_2 + 1]_{r^*}} E_j(z_1) E_j(z_2) = \frac{[u_2 - u_1]_{r^*}}{[u_2 - u_1 + 1]_{r^*}} E_j(z_2) E_j(z_1), \quad (1 \leq j \leq N), \quad (2.55)$$

$$E_i(z_1) E_j(z_2) = E_j(z_2) E_i(z_1), \quad (|i - j| \geq 2). \quad (2.56)$$

We read $E_{N+1}(z) = E_1(z)$.

The screening currents $E_j(z)$, $(1 \leq j \leq N; N \geq 3)$ satisfy the following commutation relations for $\text{Re}(r^*) < 0$.

$$\frac{1}{[u_1 - u_2 - \frac{s}{N} + 1]_{-r^*}} E_j(z_1) E_{j+1}(z_2) = \frac{1}{[u_2 - u_1 + \frac{s}{N}]_{-r^*}} E_{j+1}(z_2) E_j(z_1), \quad (1 \leq j \leq N), \quad (2.57)$$

$$\frac{[u_1 - u_2]_{-r^*}}{[u_1 - u_2 - 1]_{-r^*}} E_j(z_1) E_j(z_2) = \frac{[u_2 - u_1]_{-r^*}}{[u_2 - u_1 - 1]_{-r^*}} E_j(z_2) E_j(z_1), \quad (1 \leq j \leq N), \quad (2.58)$$

$$E_i(z_1) E_j(z_2) = E_j(z_2) E_i(z_1), \quad (|i - j| \geq 2). \quad (2.59)$$

Proposition 2.4 The screening currents $E_j(z), F_j(z)$, $(1 \leq j \leq N; N \geq 3)$ satisfy the following commutation relation $\text{Re}(r) < 0$.

$$[E_j(z_1), F_j(z_2)] = \frac{1}{x - x^{-1}} (\delta(xz_2/z_1) H_j(x^r z_2) - \delta(xz_1/z_2) H_j(x^{-r} z_2)), \quad (1 \leq j \leq N), \quad (2.60)$$

$$E_i(z_1) F_j(z_2) = F_j(z_2) E_i(z_1), \quad (1 \leq i \neq j \leq N). \quad (2.61)$$

Here we have set

$$\begin{aligned} H_j(z) &= x^{(1 - \frac{2s}{N})2j} e^{-\frac{i}{\sqrt{rr^*}} Q_{\alpha_j}} (x^{(\frac{2s}{N} - 1)j} z)^{-\frac{1}{\sqrt{rr^*}} P_{\alpha_j} + \frac{1}{rr^*}} \\ &\times : \exp \left(- \sum_{m \neq 0} \frac{1}{m} \frac{[m]}{[r^* m]} B_m^j z^{-m} \right) :, \quad (1 \leq j \leq N - 1), \end{aligned} \quad (2.62)$$

$$\begin{aligned} H_N(z) &= x^{2(N-2s)} e^{-\frac{i}{\sqrt{rr^*}} Q_{\alpha_N}} (x^{2s-N} z)^{-\frac{1}{\sqrt{rr^*}} P_{\epsilon_N} + \frac{1}{2rr^*}} z^{-\frac{1}{\sqrt{rr^*}} P_{\epsilon_1} + \frac{1}{2rr^*}} \\ &\times : \exp \left(- \sum_{m \neq 0} \frac{1}{m} \frac{[m]}{[r^* m]} B_m^N z^{-m} \right) :. \end{aligned} \quad (2.63)$$

Proposition 2.5 *The actions of η on the screenings $F_j(z)$, ($1 \leq j \leq N$; $N \geq 3$) are given by*

$$\eta(F_j(z)) = F_{j+1}(z)(x^{\frac{2s}{N}-1})^{-\sqrt{\frac{r^*}{r}}P_{\alpha_{j+1}}-\frac{r^*}{r}}, \quad (1 \leq j \leq N-2), \quad (2.64)$$

$$\eta(F_{N-1}(z)) = F_N(z)(x^{1-\frac{2s}{N}})^{\sqrt{\frac{r^*}{r}}P_{\epsilon_N}+\frac{r^*}{2r}}(x^{1-\frac{2s}{N}})^{-\sqrt{\frac{r^*}{r}}P_{\epsilon_1}+\frac{r^*}{2r}}, \quad (2.65)$$

$$\eta(F_N(z)) = F_1(z)(x^{(1-\frac{2s}{N})(N-1)})^{\sqrt{\frac{r^*}{r}}P_{\epsilon_1}+\frac{r^*}{2r}}(x^{1-\frac{2s}{N}})^{-\sqrt{\frac{r^*}{r}}P_{\epsilon_2}+\frac{r^*}{2r}}. \quad (2.66)$$

Especially we have

$$\eta(F_1(z_1)F_2(z_2)\cdots F_N(z_N)) = F_N(z_1)F_1(z_2)\cdots F_1(z_N). \quad (2.67)$$

The actions of η on the screenings $E_j(z)$, ($1 \leq j \leq N$; $N \geq 3$) are given by

$$\eta(E_j(z)) = E_{j+1}(z)(x^{\frac{2s}{N}-1})^{\sqrt{\frac{r}{r^*}}P_{\alpha_{j+1}}-\frac{r}{r^*}}, \quad (1 \leq j \leq N-2), \quad (2.68)$$

$$\eta(E_{N-1}(z)) = E_N(z)(x^{1-\frac{2s}{N}})^{-\sqrt{\frac{r}{r^*}}P_{\epsilon_N}+\frac{r}{2r^*}}(x^{1-\frac{2s}{N}})^{\sqrt{\frac{r}{r^*}}P_{\epsilon_1}+\frac{r}{2r^*}}, \quad (2.69)$$

$$\eta(E_N(z)) = E_1(z)(x^{(1-\frac{2s}{N})(N-1)})^{-\sqrt{\frac{r}{r^*}}P_{\epsilon_1}+\frac{r}{2r^*}}(x^{1-\frac{2s}{N}})^{\sqrt{\frac{r}{r^*}}P_{\epsilon_2}+\frac{r}{2r^*}}. \quad (2.70)$$

Especially we have

$$\eta(E_1(z_1)E_2(z_2)\cdots E_N(z_N)) = E_2(z_1)\cdots E_N(z_{N-1})E_1(z_N). \quad (2.71)$$

Proposition 2.6 *The screening currents $F_j(z)$, ($1 \leq j \leq N$; $N \geq 3$) and the fundamental operators $\Lambda_j(z)$, ($1 \leq j \leq N$; $N \geq 3$) commute up to delta-function $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$.*

$$[\Lambda_j(z_1), F_j(z_2)] = (-x^{r^*} + x^{-r^*})\delta\left(x^{\frac{2s}{N}j-r}\frac{z_2}{z_1}\right)\mathcal{A}_j(x^{\frac{2s}{N}j-r}z_1), \quad (1 \leq j \leq N-1), \quad (2.72)$$

$$[\Lambda_{j+1}(z_1), F_j(z_2)] = (x^{r^*} - x^{-r^*})\delta\left(x^{\frac{2s}{N}j+r}\frac{z_2}{z_1}\right)\mathcal{A}_j(x^{\frac{2s}{N}j+r}z_2), \quad (1 \leq j \leq N-1), \quad (2.73)$$

$$[\Lambda_N(z_1), F_N(z_2)] = (-x^{r^*} + x^{-r^*})\delta\left(x^{-r+2s}\frac{z_2}{z_1}\right)\mathcal{A}_N(x^{-r}z_2), \quad (2.74)$$

$$[\Lambda_1(z_1), F_N(z_2)] = (x^{r^*} - x^{-r^*})\delta\left(x^r\frac{z_2}{z_1}\right)\mathcal{A}_N(x^r z_2). \quad (2.75)$$

Here we have set

$$\begin{aligned} \mathcal{A}_j(z) &= e^{i\sqrt{\frac{r^*}{r}}Q_{\alpha_j}}x^{-\sqrt{rr^*}(P_{\epsilon_j}+P_{\epsilon_{j+1}})}(zx^{-j})^{\sqrt{\frac{r^*}{r}}P_{\alpha_j}+\frac{r^*}{r}} \\ &\times : \exp\left(\sum_{m \neq 0} \frac{1}{m}(x^{rm}\beta_m^j - x^{-rm}\beta_m^{j+1})z^{-m}\right) :, \quad (1 \leq j \leq N-1), \end{aligned} \quad (2.76)$$

$$\begin{aligned}\mathcal{A}_N(z) &= e^{i\sqrt{\frac{r^*}{r}}Q_{\alpha_N}}x^{-\sqrt{rr^*}(P_{\epsilon_N}+P_{\epsilon_1})}(zx^{2s-N})^{\sqrt{\frac{r^*}{r}}P_{\epsilon_N}+\frac{r^*}{2r}}z^{-\sqrt{\frac{r^*}{r}}P_{\epsilon_1}+\frac{r^*}{2r}} \\ &\times : \exp \left(\sum_{m \neq 0} \frac{1}{m} (x^{(r-2s)m} \beta_m^N - x^{-rm} \beta_m^1) z^{-m} \right) : .\end{aligned}\quad (2.77)$$

$$[\Lambda_j(z_1), E_j(z_2)] = (-x^r + x^{-r}) \delta \left(x^{\frac{2s}{N}j+r^*} \frac{z_2}{z_1} \right) \mathcal{B}_j(x^{\frac{2s}{N}j+r^*} z_2), \quad (1 \leq j \leq N-1), \quad (2.78)$$

$$[\Lambda_{j+1}(z_1), E_j(z_2)] = (x^r - x^{-r}) \delta \left(x^{\frac{2s}{N}j-r^*} \frac{z_2}{z_1} \right) \mathcal{B}_j(x^{\frac{2s}{N}j-r^*} z_1), \quad (1 \leq j \leq N-1), \quad (2.79)$$

$$[\Lambda_N(z_1), E_N(z_2)] = (-x^r + x^{-r}) \delta \left(x^{r^*+2s} \frac{z_2}{z_1} \right) \mathcal{B}_N(x^{r^*} z_1), \quad (2.80)$$

$$[\Lambda_1(z_1), E_N(z_2)] = (x^r - x^{-r}) \delta \left(x^{-r^*} \frac{z_2}{z_1} \right) \mathcal{B}_N(x^{-r^*} z_2). \quad (2.81)$$

Here we have set

$$\begin{aligned}\mathcal{B}_j(z) &= e^{-i\sqrt{\frac{r^*}{r}}Q_{\alpha_j}}x^{-\sqrt{rr^*}(P_{\epsilon_j}+P_{\epsilon_{j+1}})}(zx^{-j})^{-\sqrt{\frac{r^*}{r}}P_{\alpha_j}+\frac{r}{r^*}} \\ &\times : \exp \left(- \sum_{m \neq 0} \frac{1}{m} \frac{[rm]}{[r^*m]} (x^{-r^*m} \beta_m^j - x^{r^*m} \beta_m^{j+1}) z^{-m} \right) : , \quad (1 \leq j \leq N-1),\end{aligned}\quad (2.82)$$

$$\begin{aligned}\mathcal{B}_N(z) &= e^{-i\sqrt{\frac{r^*}{r}}Q_{\alpha_N}}x^{-\sqrt{rr^*}(P_{\epsilon_N}+P_{\epsilon_1})}(zx^{2s-N})^{-\sqrt{\frac{r^*}{r}}P_{\epsilon_N}+\frac{r}{2r^*}}z^{\sqrt{\frac{r^*}{r}}P_{\epsilon_1}+\frac{r}{2r^*}} \\ &\times : \exp \left(\sum_{m \neq 0} \frac{1}{m} \frac{[rm]}{[r^*m]} (x^{(-r^*-2s)m} \beta_m^N - x^{r^*m} \beta_m^1) z^{-m} \right) : .\end{aligned}\quad (2.83)$$

2.4 Comparsion with another definition

At first glance, our definition of the deformed W -algebra is different from those in [5, 6, 7]. In this section we show they are essentially the same thing. Let us set the element \mathcal{C}_m by

$$\mathcal{C}_m = \sum_{j=1}^N x^{(N-2j+1)m} \beta_m^j. \quad (2.84)$$

This element \mathcal{C}_m is η -invariant, $\eta(\mathcal{C}_m) = \mathcal{C}_m$. Let us divide $\Lambda_j(z)$ into $\Lambda_j^{DWA}(z)$ and $\mathcal{Z}(z)$.

$$\Lambda_j(z) = \Lambda_j^{DWA}(z) \mathcal{Z}(z), \quad (1 \leq j \leq N), \quad (2.85)$$

where we set

$$\Lambda_j^{DWA}(z) = x^{-2\sqrt{r(r-1)}P_{\epsilon_j}} : \exp \left(\sum_{m \neq 0} \frac{x^{rm} - x^{-rm}}{m} \left(\beta_m^j - \frac{[m]_x}{[Nm]_x} \mathcal{C}_m \right) z^{-m} \right) : \quad (2.86)$$

$$\mathcal{Z}(z) = : \exp \left(\sum_{m \neq 0} \frac{x^{rm} - x^{-rm}}{m} \frac{[m]_x}{[Nm]_x} \mathcal{C}_m z^{-m} \right) : . \quad (2.87)$$

Let us set

$$T_j^{DWA}(z) = \sum_{1 \leq s_1 < s_2 < \dots < s_j \leq N} : \Lambda_{s_1}^{DWA}(x^{-j+1}z) \Lambda_{s_2}^{DWA}(x^{-j+3}z) \dots \Lambda_{s_j}^{DWA}(x^{j-1}z) : . \quad (2.88)$$

Proposition 2.7 *The bosonic operators $T_j^{DWA}(z)$, ($1 \leq j \leq N-1$) satisfy the following relations.*

$$\begin{aligned} & f_{i,j}^{DWA}(z_2/z_1) T_i^{DWA}(z_1) T_j^{DWA}(z_2) - f_{j,i}^{DWA}(z_1/z_2) T_j^{DWA}(z_2) T_i^{DWA}(z_1) \\ &= c \sum_{k=1}^i \prod_{l=1}^{k-1} \Delta(x^{2l+1}) \times \left(\delta \left(\frac{x^{j-i+2k} z_2}{z_1} \right) f_{i-k,j+k}^{DWA}(x^{-j+i}) T_{i-k}^{DWA}(x^{-k} z_1) T_{j+k}^{DWA}(x^k z_2) \right. \\ & \quad \left. - \delta \left(\frac{x^{-j+i-2k} z_2}{z_1} \right) f_{i-k,j+k}^{DWA}(x^{j-i}) T_{i-k}^{DWA}(x^k z_1) T_{j+k}^{DWA}(x^{-k} z_2) \right), \quad (1 \leq i \leq j \leq N-1), \end{aligned} \quad (2.89)$$

where $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$. We should understand $T_N^{DWA}(z) = 1, T_j^{DWA}(z) = 0$, ($j > N$).

Here we set the constant c and the auxiliary function $\Delta(z)$ in (2.17). Here we set the structure functions,

$$\begin{aligned} & f_{i,j}^{DWA}(z) = f_{i,j}(z)|_{s=N} \\ &= \exp \left(\sum_{m=1}^{\infty} \frac{1}{m} (1 - x^{2rm}) (1 - x^{-2(r-1)m}) \frac{(1 - x^{2m \min(i,j)}) (1 - x^{2m(N - \max(i,j))})}{(1 - x^{2m}) (1 - x^{2Nm})} x^{|i-j|m} z^m \right). \end{aligned} \quad (2.90)$$

Proposition 2.8 *The operators $T_j^{DWA}(z)$ and $\mathcal{Z}(z)$ commutes with each other.*

$$T_j^{DWA}(z_1) \mathcal{Z}(z_2) = \mathcal{Z}(z_2) T_j^{DWA}(z_1), \quad (1 \leq j \leq N-1). \quad (2.91)$$

Therefore three parameter deformed W -algebra $T_j(z)$ is realized as an extension of two parameter deformed W -algebra $T_j^{DWA}(z)$ in [5, 6, 7]. Note that upon the specialization $s = N$ we have

$$[B_n^N, B_m^N] = 0, \quad [B_m^j, B_n^N] = 0 \text{ for } j \neq N. \quad (2.92)$$

Hence we can regard $B_m^N = 0$ and $T_j(z) = T_j^{DWA}(z)$, $T_N^{DWA}(z) = 1$.

3 Local Integrals of Motion

In this section we construct the local integrals of motion \mathcal{I}_n . We study the generic case : $0 < x < 1$, $r \in \mathbb{C}$ and $\text{Re}(s) > 0$.

3.1 Local Integrals of Motion for $W_{q,t}(\widehat{sl}_N)$

Let us set the function $h(u)$ and $h^*(u)$ by

$$h(u) = \frac{[u]_s[u+r]_s}{[u+1]_s[u+r^*]_s}, \quad h^*(u) = \frac{[u]_s[u-r^*]_s}{[u+1]_s[u-r]_s}, \quad (3.1)$$

where we have set $z = x^{2u}$.

Definition 3.1

- We define \mathcal{I}_n for regime $\text{Re}(s) > 2$ and $\text{Re}(r^*) < 0$ by

$$\mathcal{I}_n = \int \cdots \int_C \prod_{j=1}^n \frac{dz_j}{2\pi\sqrt{-1}z_j} \prod_{1 \leq j < k \leq n} h(u_k - u_j) T_1(z_1) \cdots T_1(z_n) \quad (n = 1, 2, \dots). \quad (3.2)$$

Here, the contour C encircles $z_j = 0$ in such a way that $z_j = x^{-2+2sl}z_k, x^{-2r^*+2sl}z_k$ ($l = 0, 1, 2, \dots$) is inside and $z_j = x^{2-2sl}z_k, x^{2r^*-2sl}z_k$ ($l = 0, 1, 2, \dots$) is outside for $1 \leq j < k \leq n$. We call \mathcal{I}_n the local integrals of motion for the deformed W -algebra. The definitions of \mathcal{I}_n for generic $\text{Re}(s) > 0$ and $r \in \mathbb{C}$ should be understood as analytic continuation.

- We define \mathcal{I}_n^* for regime $\text{Re}(s) > 2$ and $\text{Re}(r) > 0$ by

$$\mathcal{I}_n^* = \int \cdots \int_C \prod_{j=1}^n \frac{dz_j}{2\pi\sqrt{-1}z_j} \prod_{1 \leq j < k \leq n} h^*(u_k - u_j) T_1(z_1) \cdots T_1(z_n) \quad (n = 1, 2, \dots). \quad (3.3)$$

Here, the contour C encircles $z_j = 0$ in such a way that $z_j = x^{-2+2sl}z_k, x^{2r+2sl}z_k$ ($l = 0, 1, 2, \dots$) is inside and $z_j = x^{2-2sl}z_k, x^{-2r-2sl}z_k$ ($l = 0, 1, 2, \dots$) is outside for $1 \leq j < k \leq n$. We call \mathcal{I}_n^* the local integrals of motion for the deformed W -algebra. The definitions of \mathcal{I}_n^* for generic $\text{Re}(s) > 0$ and $r \in \mathbb{C}$ should be understood as analytic continuation.

The following is one of **Main Results** of this paper.

Theorem 3.1 *The local integrals of motion \mathcal{I}_n commute with each other*

$$[\mathcal{I}_n, \mathcal{I}_m] = 0 \quad (m, n = 1, 2, \dots). \quad (3.4)$$

The local integrals of motion \mathcal{I}_n^ commute with each other*

$$[\mathcal{I}_n^*, \mathcal{I}_m^*] = 0 \quad (m, n = 1, 2, \dots). \quad (3.5)$$

3.2 Laurent-Series Formulae

In this subsection we prepare another formulae of the local integrals of motion \mathcal{I}_n . Because the integral contour of the definition of the local integrals of motion \mathcal{I}_n is not annulus. *i.e.* $|x^{-p}z_k| < |z_j| < |x^p z_k|$, the defining relations of the deformed W -algebra (2.16) should be used carefully. Hence, in order to show the commutation relations $[\mathcal{I}_m, \mathcal{I}_n] = 0$, it is better for us to deform the integral representations of the local integrals of motion \mathcal{I}_n to another formulae, in which the defining relations of the deformed W -algebra (2.16) can be used safely.

Let us set the auxiliary function $s(z), s^*(z)$ by $h(u) = s(z)f_{11}(z)$, $h^*(u) = s^*(z)f_{11}(z)$, ($z = x^{2u}$) where $h(u), h^*(u)$ and $f_{11}(z)$ are given in the previous section. We have explicitly

$$s(z) = x^{-2r^*} \frac{(z; x^{2s})_\infty (x^{2s-2r}z; x^{2s})_\infty}{(x^{2s-2}z; x^{2s})_\infty (x^{-2r^*}z; x^{2s})_\infty} \times \frac{(1/z; x^{2s})_\infty (x^{2s-2r}/z; x^{2s})_\infty}{(x^{2s-2}/z; x^{2s})_\infty (x^{-2r^*}/z; x^{2s})_\infty}, \quad (3.6)$$

$$s^*(z) = x^{-2r^*} \frac{(z; x^{2s})_\infty (x^{2s+2r^*}z; x^{2s})_\infty}{(x^{2s-2}z; x^{2s})_\infty (x^{2r}z; x^{2s})_\infty} \times \frac{(1/z; x^{2s})_\infty (x^{2s+2r^*}/z; x^{2s})_\infty}{(x^{2s-2}/z; x^{2s})_\infty (x^{2r}/z; x^{2s})_\infty}. \quad (3.7)$$

Let us set the auxiliary functions $g_{i,j}(z)$ by fusion procedure

$$\begin{aligned} g_{i,1}(z) &= g_{1,1}(x^{-i+1}z)g_{1,1}(x^{-i+3}z) \cdots g_{1,1}(x^{i-1}z), \\ g_{i,j}(z) &= g_{i,1}(x^{-j+1}z)g_{i,1}(x^{-j+3}z) \cdots g_{i,1}(x^{j-1}z). \end{aligned} \quad (3.8)$$

where $g_{11}(z) = f_{11}(z)$ is the structure function of the deformed W -algebra defined in (2.16).

$$f_{1,1}(z) = \frac{1}{1-z} \frac{(x^{2s-2}z; x^{2s})_\infty (x^{2r}z; x^{2s})_\infty (x^{-2r^*}z; x^{2s})_\infty}{(x^2z; x^{2s})_\infty (x^{2r^*+2s}z; x^{2s})_\infty (x^{2s-2r}z; x^{2s})_\infty}. \quad (3.9)$$

The structure functions $f_{1,j}(z)$ and $g_{1,j}(z)$ have the following relations

$$g_{1,j}(z) = \Delta(x^{-j+2}z)\Delta(x^{-j+4}z) \cdots \Delta(x^{j-2}z)f_{1,j}(z). \quad (3.10)$$

Here $\Delta(z)$ is given by $\Delta(z) = \frac{(1-x^{r+r^*}z)(1-x^{-r-r^*}z)}{(1-xz)(1-x^{-1}z)}$.

Let us set the formal power series $\mathcal{A}(z_1, z_2, \dots, z_n)$ by

$$\mathcal{A}(z_1, z_2, \dots, z_n) = \sum_{k_1, \dots, k_n \in \mathbb{Z}} a_{k_1, \dots, k_n} z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}. \quad (3.11)$$

We define the symbol $[\cdot \cdot \cdot]_{1, z_1 \dots z_n}$ by

$$[\mathcal{A}(z_1, z_2, \dots, z_n)]_{1, z_1 \dots z_n} = a_{0,0,\dots,0}. \quad (3.12)$$

Let us set $D = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid \sum_{k_1, \dots, k_n \in \mathbb{Z}} |a_{k_1, \dots, k_n} z_1^{k_1} z_2^{k_2} \dots z_n^{k_n}| < +\infty\}$. When we assume closed curve J is contained in D , we have

$$[\mathcal{A}(z_1, z_2, \dots, z_n)]_{1, z_1 \dots z_n} = \int \dots \int_J \prod_{j=1}^n \frac{dz_j}{2\pi\sqrt{-1}z_j} \mathcal{A}(z_1, z_2, \dots, z_n). \quad (3.13)$$

Let us set the auxiliary functions, $s_{11}(z) = s(z)$, $h_{11}(z) = h(u)$, ($z = x^{2u}$) and

$$\begin{aligned} s_{i,1}(z) &= s_{1,1}(x^{-i+1}z) s_{1,1}(x^{-i+3}z) \dots s_{1,1}(x^{i-1}z), \\ s_{i,j}(z) &= s_{i,1}(x^{-j+1}z) s_{i,1}(x^{-j+3}z) \dots s_{i,1}(x^{j-1}z), \end{aligned} \quad (3.14)$$

$$\begin{aligned} h_{i,1}(z) &= h_{1,1}(x^{-i+1}z) h_{1,1}(x^{-i+3}z) \dots h_{1,1}(x^{i-1}z), \\ h_{i,j}(z) &= h_{i,1}(x^{-j+1}z) h_{i,1}(x^{-j+3}z) \dots h_{i,1}(x^{j-1}z), \end{aligned} \quad (3.15)$$

and

$$\begin{aligned} s_{i,1}^*(z) &= s_{1,1}^*(x^{-i+1}z) s_{1,1}^*(x^{-i+3}z) \dots s_{1,1}^*(x^{i-1}z), \\ s_{i,j}^*(z) &= s_{i,1}^*(x^{-j+1}z) s_{i,1}^*(x^{-j+3}z) \dots s_{i,1}^*(x^{j-1}z), \end{aligned} \quad (3.16)$$

$$\begin{aligned} h_{i,1}^*(z) &= h_{1,1}^*(x^{-i+1}z) h_{1,1}^*(x^{-i+3}z) \dots h_{1,1}^*(x^{i-1}z), \\ h_{i,j}^*(z) &= h_{i,1}^*(x^{-j+1}z) h_{i,1}^*(x^{-j+3}z) \dots h_{i,1}^*(x^{j-1}z). \end{aligned} \quad (3.17)$$

In what follows we use the notation of the ordered product

$$\prod_{\overrightarrow{l \in L}} T_1(z_l) = T_1(z_{l_1}) T_1(z_{l_2}) \dots T_1(z_{l_n}), \quad (L = \{l_1, \dots, l_n \mid l_1 < l_2 < \dots < l_n\}). \quad (3.18)$$

Theorem 3.2 *For $\text{Re}(s) > N$ and $\text{Re}(r^*) < 0$, the local integrals of motion \mathcal{I}_n are written as*

$$\mathcal{I}_n = \left[\prod_{1 \leq j < k \leq n} s(z_k/z_j) \mathcal{O}_n(z_1, z_2, \dots, z_n) \right]_{1, z_1 \dots z_n}. \quad (3.19)$$

For $\text{Re}(s) > N$ and $\text{Re}(r) > 0$, the local integrals of motion \mathcal{I}_n^ are written as*

$$\mathcal{I}_n^* = \left[\prod_{1 \leq j < k \leq n} s^*(z_k/z_j) \mathcal{O}_n(z_1, z_2, \dots, z_n) \right]_{1, z_1 \dots z_n}. \quad (3.20)$$

Here we set the operator $\mathcal{O}_n(z_1, z_2, \dots, z_n)$ by

$$\begin{aligned}
\mathcal{O}_n(z_1, z_2, \dots, z_n) = & \sum_{\substack{\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_N \geq 0 \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 + \dots + N\alpha_N = n}} \sum_{\substack{\{A_j^{(s)}\}_{j=1, \dots, N} \\ A_j^{(s)} \subset \{1, 2, \dots, n\}, \quad |A_j^{(s)}| = s, \quad \cup_{s=1}^N \cup_{j=1}^{\alpha_s} A_j^{(s)} = \{1, 2, \dots, n\} \\ \text{Min}(A_1^{(s)}) < \text{Min}(A_2^{(s)}) < \dots < \text{Min}(A_{\alpha_s}^{(s)})}} \\
& \times \prod_{j \in \overline{A_{Min}^{(1)}}} T_1(z_j) \prod_{j \in \overline{A_{Min}^{(2)}}} T_2(x^{-1}z_j) \dots \prod_{j \in \overline{A_{Min}^{(t)}}} T_t(x^{-1+t-2[\frac{t}{2}]}z_j) \dots \prod_{j \in \overline{A_{Min}^{(N)}}} T_N(x^{-1+N-2[\frac{N}{2}]}z_j) \\
& \times \prod_{t=1}^N \left((-c)^{t-1} \prod_{u=1}^{t-1} \Delta(x^{2u+1})^{t-u-1} \right)^{\alpha_t} \prod_{t=1}^N \prod_{\substack{j=1 \\ j_1=A_{j,1}^{(t)} \\ \dots \\ j_t=A_{j,t}^{(t)}}}^{\alpha_t} \sum_{\substack{\sigma \in S_t \\ \sigma(1)=1}} \prod_{\substack{u=1 \\ u \neq [\frac{t}{2}]+1}}^t \delta \left(\frac{x^2 z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}} \right) \\
& \times \prod_{t=1}^N \prod_{\substack{j < k \\ j, k \in A_{Min}^{(t)}}} g_{t,t} \left(\frac{z_k}{z_j} \right) \prod_{1 \leq t < u \leq N} \prod_{\substack{j \in A_{Min}^{(t)} \\ k \in A_{Min}^{(u)}}} g_{t,u} \left(x^{u-t-2[\frac{u}{2}]+2[\frac{t}{2}]} \frac{z_k}{z_j} \right). \tag{3.21}
\end{aligned}$$

Here we have set the constant c and the function $\Delta(z)$ in (2.17). When the index set $A_j^{(t)} = \{j_1, j_2, \dots, j_t | j_1 < j_2 < \dots < j_t\}$, ($1 \leq t \leq N, 1 \leq j \leq \alpha_t$), we set $A_{j,k}^{(t)} = j_k$, and $A_{Min}^{(t)} = \{A_{1,1}^{(t)}, A_{2,1}^{(t)}, \dots, A_{\alpha_t,1}^{(t)}\}$. Here we should understand $z_{j_{\sigma(t+1)}} = z_{j_{\sigma(1)}}$ in the delta-function $\delta \left(\frac{x^2 z_{j_{\sigma(t+1)}}}{z_{j_{\sigma(t)}}} \right)$.

Example We summarize the operators \mathcal{O}_n very explicitly.

$$\mathcal{O}_1(z) = T_1(z), \tag{3.22}$$

$$\mathcal{O}_2(z_1, z_2) = g_{1,1}(z_2/z_1)T_1(z_1)T_1(z_2) - c\delta(x^2 z_2/z_1)T_2(x^{-1}z_1), \tag{3.23}$$

$$\begin{aligned}
\mathcal{O}_3(z_1, z_2, z_3) = & g_{11}(z_2/z_1)g_{1,1}(z_3/z_1)g_{1,1}(z_3/z_2)T_1(z_1)T_1(z_2)T_1(z_3) \\
& - cg_{1,2}(x^{-1}z_2/z_1)T_1(z_1)\delta(x^2 z_3/z_2)T_2(x^{-1}z_2) \\
& - cg_{1,2}(x^{-1}z_1/z_2)T_1(z_2)\delta(x^2 z_3/z_1)T_2(x^{-1}z_1) \\
& - cg_{1,2}(x^{-1}z_1/z_3)T_1(z_3)\delta(x^2 z_2/z_1)T_2(x^{-1}z_1) \\
& + c^2 \Delta(x^3)(\delta(x^2 z_2/z_1)\delta(x^2 z_1/z_3) + \delta(x^2 z_1/z_2)\delta(x^2 z_3/z_1))T_3(z_1) \tag{3.24}
\end{aligned}$$

$$\begin{aligned}
\mathcal{O}_4(z_1, z_2, z_3, z_4) = & \prod_{1 \leq j < k \leq 4} g_{11}(z_k/z_j)T_1(z_1)T_1(z_2)T_1(z_3)T_1(z_4) \\
& - cg_{11}(z_2/z_1)g_{12}(x^{-1}z_3/z_1)g_{12}(x^{-1}z_3/z_2)T_1(z_1)T_1(z_2)T_2(x^{-1}z_3)\delta(x^2 z_4/z_3)
\end{aligned}$$

$$\begin{aligned}
& - cg_{11}(z_3/z_1)g_{12}(x^{-1}z_2/z_1)g_{12}(x^{-1}z_2/z_3)T_1(z_1)T_1(z_3)T_2(x^{-1}z_2)\delta(x^2z_4/z_2) \\
& - cg_{11}(z_4/z_1)g_{12}(x^{-1}z_2/z_1)g_{12}(x^{-1}z_2/z_4)T_1(z_1)T_1(z_4)T_2(x^{-1}z_2)\delta(x^2z_3/z_2) \\
& - cg_{11}(z_3/z_2)g_{12}(x^{-1}z_1/z_2)g_{12}(x^{-1}z_1/z_3)T_1(z_2)T_1(z_3)T_2(x^{-1}z_1)\delta(x^2z_4/z_1) \\
& - cg_{11}(z_4/z_2)g_{12}(x^{-1}z_1/z_2)g_{12}(x^{-1}z_1/z_4)T_1(z_2)T_1(z_4)T_2(x^{-1}z_1)\delta(x^2z_3/z_1) \\
& - cg_{11}(z_4/z_3)g_{12}(x^{-1}z_1/z_3)g_{12}(x^{-1}z_1/z_4)T_1(z_3)T_1(z_4)T_2(x^{-1}z_1)\delta(x^2z_2/z_1) \\
& + c^2g_{22}(z_3/z_1)\delta(x^2z_2/z_1)\delta(x^2z_4/z_3)T_2(x^{-1}z_1)T_2(x^{-1}z_3) \\
& + c^2g_{22}(z_2/z_1)\delta(x^2z_3/z_1)\delta(x^2z_4/z_2)T_2(x^{-1}z_1)T_2(x^{-1}z_2) \\
& + c^2g_{22}(z_2/z_1)\delta(x^2z_4/z_1)\delta(x^2z_3/z_2)T_2(x^{-1}z_1)T_2(x^{-1}z_2) \\
& + c^2\Delta(x^3)g_{13}(z_2/z_1)T_1(z_1)T_3(z_2)(\delta(x^2z_2/z_3)\delta(x^2z_4/z_2) + \delta(x^2z_2/z_4)\delta(x^2z_3/z_2)) \\
& + c^2\Delta(x^3)g_{13}(z_1/z_2)T_1(z_2)T_3(z_1)(\delta(x^2z_1/z_3)\delta(x^2z_4/z_1) + \delta(x^2z_1/z_4)\delta(x^2z_3/z_1)) \\
& + c^2\Delta(x^3)g_{13}(z_1/z_3)T_1(z_3)T_3(z_1)(\delta(x^2z_1/z_2)\delta(x^2z_4/z_1) + \delta(x^2z_1/z_4)\delta(x^2z_2/z_1)) \\
& + c^2\Delta(x^3)g_{13}(z_1/z_4)T_1(z_4)T_3(z_1)(\delta(x^2z_1/z_2)\delta(x^2z_3/z_1) + \delta(x^2z_1/z_3)\delta(x^2z_2/z_1)) \\
& - c^3\Delta(x^3)\Delta(x^5)^2T_4(x^{-1}z_1) \\
& \times (\delta(x^2z_1/z_2)\delta(x^2z_3/z_1)\delta(x^2z_4/z_3) + \delta(x^2z_1/z_2)\delta(x^2z_4/z_1)\delta(x^2z_3/z_4) \\
& + \delta(x^2z_1/z_3)\delta(x^2z_2/z_1)\delta(x^2z_4/z_2) + \delta(x^2z_1/z_3)\delta(x^2z_4/z_1)\delta(x^2z_2/z_4) \\
& + \delta(x^2z_1/z_4)\delta(x^2z_2/z_1)\delta(x^2z_3/z_2) + \delta(x^2z_1/z_4)\delta(x^2z_2/z_1)\delta(x^2z_2/z_3))
\end{aligned}$$

We should understand above as $T_j(z) = 0$ for $j > N$.

3.3 Weakly sense equality

In order to show theorem, we introduce a “weak sense” equality

Definition 3.2 *We say the operators $\mathcal{P}(z_1, z_2, \dots, z_n)$ and $\mathcal{Q}(z_1, z_2, \dots, z_n)$ are equal in the “weak sense” if*

$$\prod_{1 \leq i < j \leq n} s(z_j/z_i) \mathcal{P}(z_1, z_2, \dots, z_n) = \prod_{1 \leq i < j \leq n} s(z_j/z_i) \mathcal{Q}(z_1, z_2, \dots, z_n). \quad (3.25)$$

We write $\mathcal{P}(z_1, z_2, \dots, z_n) \sim \mathcal{Q}(z_1, z_2, \dots, z_n)$, showing the weak equality.

For example $\delta(z_1/z_2) \sim 0$ and $\frac{1}{z_1 - z_2} \delta(z_1/z_2) \sim 0$.

Proposition 3.3 *The following relations hold in the weak sense for $1 \leq j \leq N$*

$$\begin{aligned}
& \left(g_{1,j} \left(\frac{x^{-1+j-2[\frac{j}{2}]w_1}}{z_1} \right) T_1(z_1) T_j(x^{-1+j-2[\frac{j}{2}]w_1}) - g_{j,1} \left(\frac{x^{1-j+2[\frac{j}{2}]z_1}}{w_1} \right) T_j(x^{-1+j-2[\frac{j}{2}]w_1}) T_1(z_1) \right) \\
& \times \sum_{\substack{\sigma \in S_j \\ \sigma(1)=1}} \prod_{\substack{t=1 \\ t \neq [\frac{j}{2}]+1}}^j \delta \left(\frac{x^2 w_{\sigma(t+1)}}{w_{\sigma(t)}} \right) \sim c \prod_{t=1}^{j-1} \Delta(x^{2t+1}) \sum_{\substack{\sigma \in S_j \\ \sigma(1)=1}} \prod_{\substack{t=1 \\ t \neq [\frac{j}{2}]+1}}^j \delta \left(\frac{x^2 w_{\sigma(t+1)}}{w_{\sigma(t)}} \right) \\
& \times \left(\delta \left(\frac{x^{2j-2[\frac{j}{2}]w_1}}{z_1} \right) T_{j+1}(x^{j-2[\frac{j}{2}]w_1}) - \delta \left(\frac{x^{-2-2[\frac{j}{2}]w_1}}{z_1} \right) T_{j+1}(x^{j-2-2[\frac{j}{2}]w_1}) \right). \tag{3.26}
\end{aligned}$$

We should understand $T_{N+1}(z) = 0$ and $w_{\sigma(j+1)} = w_{\sigma(1)}$ in the delta-function $\delta \left(\frac{x^2 w_{\sigma(j+1)}}{w_{\sigma(j)}} \right)$.

Proof We explain the mechanism by the simplest case for $N \geq 3$.

$$\begin{aligned}
& (g_{1,2}(x^{-1}z_2/z_1)T_1(z_1)T_2(x^{-1}z_2) - g_{2,1}(xz_1/z_2)T_2(x^{-1}z_2)T_1(z_1)) \delta(x^2z_3/z_2) \\
& = g_{1,2}(x^{-1}z_2/z_1)c(\delta(z_2/z_1) - \delta(x^{-2}z_2/z_1))\delta(x^2z_3/z_2)T_1(z_1)T_2(x^{-1}z_2) \\
& + \Delta(xz_1/z_2)\delta(x^2z_3/z_2)(f_{1,2}(x^{-1}z_2/z_1)T_1(z_1)T_2(x^{-1}z_2) - f_{2,1}(xz_1/z_2)T_2(x^{-1}z_2)T_1(z_1)). \tag{3.27}
\end{aligned}$$

Here we have used $g_{1,2}(z) = \Delta(z)f_{1,2}(z)$ and

$$\Delta(z) - \Delta(z^{-1}) = c(\delta(xz) - \delta(x^{-1}z)), \quad \Delta(z) = \frac{(1 - x^{2r-1}z)(1 - x^{-2r+1}z)}{(1 - xz)(1 - x^{-1}z)}. \tag{3.28}$$

Using $\delta(z_1/z_2) \sim 0$ and $\delta(x^2z_1/z_2)\delta(x^2z_3/z_2) \sim 0$, $\Delta(x^3) = \Delta(x^{-3})$, and the defining relation of the deformed W -algebra (2.16), we get this proposition. Q.E.D.

As the same manner as above, we have the following proposition.

Proposition 3.4 *The following relations hold in the weak sense for $i, j \geq 2$*

$$\begin{aligned}
& g_{i,j} \left(\frac{x^{j-i-2[\frac{j-i}{2}]w_1}}{z_1} \right) T_i(x^{-1+i-2[\frac{i}{2}]z_1}) T_j(x^{-1+j-2[\frac{j}{2}]w_1}) \\
& \times \sum_{\substack{\sigma \in S_j \\ \sigma(1)=1}} \prod_{\substack{t=1 \\ t \neq [\frac{j}{2}]+1}}^i \delta \left(\frac{x^2 z_{\sigma(t+1)}}{z_{\sigma(t)}} \right) \sum_{\substack{\sigma \in S_j \\ \sigma(1)=1}} \prod_{\substack{t=1 \\ t \neq [\frac{j}{2}]+1}}^j \delta \left(\frac{x^2 w_{\sigma(t+1)}}{w_{\sigma(t)}} \right) \\
& \sim g_{j,i} \left(\frac{x^{i-j-2[\frac{i-j}{2}]z_1}}{w_1} \right) T_j(x^{-1+j-2[\frac{j}{2}]w_1}) T_i(x^{-1+i-2[\frac{i}{2}]z_1}) \\
& \times \sum_{\substack{\sigma \in S_j \\ \sigma(1)=1}} \prod_{\substack{t=1 \\ t \neq [\frac{j}{2}]+1}}^i \delta \left(\frac{x^2 z_{\sigma(t+1)}}{z_{\sigma(t)}} \right) \sum_{\substack{\sigma \in S_j \\ \sigma(1)=1}} \prod_{\substack{t=1 \\ t \neq [\frac{j}{2}]+1}}^j \delta \left(\frac{x^2 w_{\sigma(t+1)}}{w_{\sigma(t)}} \right). \tag{3.29}
\end{aligned}$$

Example The operators $T_1(z), T_2(z), T_3(z)$ satisfy

$$\begin{aligned} & g_{1,1} \left(\frac{z_2}{z_1} \right) T_1(z_1)T_1(z_2) - g_{1,1} \left(\frac{z_1}{z_2} \right) T_1(z_2)T_1(z_1) \\ & \sim c \left(T_2(x^{-1}z_1)\delta \left(\frac{x^2z_2}{z_1} \right) - T_2(xz_1)\delta \left(\frac{x^2z_1}{z_2} \right) \right), \end{aligned} \quad (3.30)$$

$$\begin{aligned} & \left(g_{1,2} \left(\frac{x^{-1}z_2}{z_1} \right) T_1(z_1)T_2(x^{-1}z_2) - g_{2,1} \left(\frac{xz_1}{z_2} \right) T_2(x^{-1}z_2)T_1(z_1) \right) \delta \left(\frac{x^2z_3}{z_2} \right) \\ & \sim c\Delta(x^3) \left(T_3(z_2)\delta \left(\frac{x^2z_2}{z_1} \right) - T_3(z_3)\delta \left(\frac{x^2z_1}{z_3} \right) \right) \delta \left(\frac{x^2z_3}{z_2} \right), \end{aligned} \quad (3.31)$$

$$\begin{aligned} & \left(g_{2,2} \left(\frac{w_1}{z_1} \right) T_2(z_1)T_2(w_1) - g_{2,2} \left(\frac{z_1}{w_1} \right) T_2(w_1)T_2(z_1) \right) \\ & \times \delta \left(\frac{x^2z_1}{z_2} \right) \delta \left(\frac{x^2w_1}{w_2} \right) \sim 0, \end{aligned} \quad (3.32)$$

$$\begin{aligned} & \left(g_{1,3} \left(\frac{w_1}{z_1} \right) T_1(z_1)T_3(w_1) - g_{3,1} \left(\frac{z_1}{w_1} \right) T_3(w_1)T_1(z_1) \right) \\ & \times \left(\delta \left(\frac{x^2w_1}{w_2} \right) \delta \left(\frac{x^2w_3}{w_1} \right) + \delta \left(\frac{x^2w_1}{w_3} \right) \delta \left(\frac{x^2w_2}{w_1} \right) \right) \sim 0, \end{aligned} \quad (3.33)$$

$$\begin{aligned} & \left(g_{2,3} \left(\frac{xw_1}{z_1} \right) T_2(x^{-1}z_1)T_3(w_1) - g_{3,2} \left(\frac{x^{-1}z_1}{z_2} \right) T_3(z_2)T_2(x^{-1}z_1) \right) \\ & \times \delta \left(\frac{x^2z_1}{z_2} \right) \left(\delta \left(\frac{x^2w_1}{w_2} \right) \delta \left(\frac{x^2w_3}{w_1} \right) + \delta \left(\frac{x^2w_1}{w_3} \right) \delta \left(\frac{x^2w_2}{w_1} \right) \right) \sim 0, \end{aligned} \quad (3.34)$$

$$\begin{aligned} & \left(g_{3,3} \left(\frac{w_1}{z_1} \right) T_3(z_1)T_3(w_1) - g_{3,3} \left(\frac{w_1}{z_1} \right) T_3(w_1)T_3(z_1) \right) \\ & \times \left(\delta \left(\frac{x^2w_1}{w_2} \right) \delta \left(\frac{x^2w_3}{w_1} \right) + \delta \left(\frac{x^2w_1}{w_3} \right) \delta \left(\frac{x^2w_2}{w_1} \right) \right) \\ & \times \left(\delta \left(\frac{x^2w_1}{w_2} \right) \delta \left(\frac{x^2w_3}{w_1} \right) + \delta \left(\frac{x^2w_1}{w_3} \right) \delta \left(\frac{x^2w_2}{w_1} \right) \right) \sim 0. \end{aligned} \quad (3.35)$$

We should understand $T_j(z) = 0$ for $j > N$.

Let us introduce S_n -invariance in the “weak sense”.

Definition 3.3 We call the operator $\mathcal{P}(z_1, z_2, \dots, z_n)$ is S_n -invariant in the “weak sense” if

$$\mathcal{P}(z_1, z_2, \dots, z_n) \sim \mathcal{P}(z_{\sigma(1)}, z_{\sigma(2)}, \dots, z_{\sigma(n)}), \quad (\sigma \in S_n). \quad (3.36)$$

Example The operator $\mathcal{O}_2(z_1, z_2) = g_{11}(z_2/z_1)T_1(z_1)T_1(z_2) - c\delta(x^2z_2/z_1)T_2(x^{-1}z_1)$ is S_2 -invariant.

Theorem 3.5 *The operator \mathcal{O}_n defined in Theorem 3.2 is S_n -invariant in the weak sense.*

$$\mathcal{O}_n(z_1, z_2, \dots, z_n) \sim \mathcal{O}_n(z_{\sigma(1)}, z_{\sigma(2)}, \dots, z_{\sigma(n)}) \quad (\sigma \in S_n). \quad (3.37)$$

This theorem plays an important role in proof of the main theorem 3.1. We will show above theorem in the next section.

3.4 Proof of S_n -Invariance for $\mathcal{O}_n(z_1, \dots, z_n)$

In this section we give proof of theorem 3.5. Proof for special case \widehat{sl}_2 is summarized in [13]. By straightforward calculations we have the following proposition.

Proposition 3.6 *The following relation holds in weakly sense.*

$$\begin{aligned} & \prod_{1 \leq j < k \leq M} g_{1,1}(z_k/z_j) \prod_{1 \leq j \leq M} T_1(z_j) - (z_1 \leftrightarrow z_2) \\ & \sim \sum_{t=0}^M \sum_{3 \leq j_3 < j_4 < \dots < j_{t+2} \leq M} (-1)^t c^{t+1} \prod_{u=1}^t \Delta(x^{2u+1})^{t+1-u} \\ & \times \prod_{\substack{3 \leq j < k \leq M \\ j, k \neq j_3, \dots, j_{t+2}}} g_{1,1}\left(\frac{z_k}{z_j}\right) \prod_{\substack{3 \leq j \leq M \\ j \neq j_3, \dots, j_{t+2}}} g_{1,t+2}\left(x^{-1+t-2[\frac{t}{2}]} \frac{z_1}{z_j}\right) \prod_{\substack{3 \leq j \leq M \\ j \neq j_3, \dots, j_{t+2}}} T_1(z_j) \\ & \times T_{t+2}(x^{-1+t-2[\frac{t}{2}]} z_1) \sum_{\substack{\sigma \in S_{t+2} \\ \sigma(1)=1}} \prod_{\substack{u=1 \\ u \neq [\frac{t}{2}]+2 \\ j_1=1, j_2=2}}^{t+2} \delta\left(\frac{x^2 z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}}\right) - (z_1 \leftrightarrow z_2). \end{aligned} \quad (3.38)$$

We should understand $T_j(z) = 0$ ($j > N$).

Proof of Theorem 3.5 At first we consider \widehat{sl}_3 case for reader's convenience. The operator \mathcal{O}_n for \widehat{sl}_3 is written very explicitly as following.

$$\begin{aligned} & \mathcal{O}_n(z_1, z_2, z_3, \dots, z_n) \\ & = \sum_{\substack{\alpha_1, \alpha_2, \alpha_3 \geq 0 \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 = n}} (-c)^{\alpha_2 + 2\alpha_3} \Delta(x^3)^{\alpha_3} \sum_{\substack{\{A_j^{(s)}\}_{s=1, \dots, 3} \\ j=1, \dots, \alpha_s \\ A_j^{(s)} \subset \{1, 2, \dots, n\}, |A_j^{(s)}| = s, \cup_{s=1}^3 \cup_{j=1}^{\alpha_s} A_j^{(s)} = \{1, 2, \dots, n\} \\ \text{Min}(A_1^{(s)}) < \text{Min}(A_2^{(s)}) < \dots < \text{Min}(A_{\alpha_s}^{(s)})}} \end{aligned}$$

$$\begin{aligned}
& \times \prod_{j \in \overrightarrow{A_{Min}^{(1)}}} T_1(z_j) \prod_{j \in \overrightarrow{A_{Min}^{(2)}}} T_2(x^{-1}z_j) \prod_{j \in \overrightarrow{A_{Min}^{(3)}}} T_3(z_j) \\
& \times \prod_{\substack{j=1 \\ j_1=A_{j,1}^{(2)} \\ j_2=A_{j,2}^{(2)}}}^{\alpha_2} \delta\left(\frac{x^2 z_{j_2}}{z_{j_1}}\right) \prod_{\substack{j=1 \\ j_1=A_{j,1}^{(3)} \\ j_2=A_{j,2}^{(3)} \\ j_3=A_{j,3}^{(3)}}}^{\alpha_3} \left(\delta\left(\frac{x^2 z_{j_2}}{z_{j_1}}\right) \delta\left(\frac{x^2 z_{j_1}}{z_{j_3}}\right) + \delta\left(\frac{x^2 z_{j_3}}{z_{j_1}}\right) \delta\left(\frac{x^2 z_{j_1}}{z_{j_2}}\right) \right) \\
& \times \prod_{t=1}^3 \prod_{\substack{1 \leq j < k \leq n \\ j, k \in A_{Min}^{(t)}}} g_{t,t} \left(\frac{z_k}{z_j} \right) \prod_{\substack{j \in A_{Min}^{(1)} \\ k \in A_{Min}^{(2)}}} g_{1,2} \left(\frac{x^{-1} z_k}{z_j} \right) \prod_{\substack{j \in A_{Min}^{(1)} \\ k \in A_{Min}^{(3)}}} g_{1,3} \left(\frac{z_k}{z_j} \right) \prod_{\substack{j \in A_{Min}^{(2)} \\ k \in A_{Min}^{(3)}}} g_{2,3} \left(\frac{x z_k}{z_j} \right).
\end{aligned} \tag{3.39}$$

In order to show S_n -invariance, it is enough to show the case of the permutations $\sigma = (i, i+1)$ for $1 \leq i \leq n-1$. Let us study the permutation $\sigma = (i, i+1)$. Because of the cancellations, the difference $\mathcal{O}_n(\cdots, z_i, z_{i+1}, \cdots) - \mathcal{O}_n(\cdots, z_{i+1}, z_i, \cdots)$ has simplification. We don't have to consider every summation $\sum_{\substack{\{A_j^{(s)}\} \\ s=1, \dots, 3 \\ j=1, \dots, \alpha_s}}$ in the definition of \mathcal{O}_n . We only have to consider the summation of the following three cases for any $\sigma = (i, i+1)$

- (1) $\{i, i+1\} \subset \cup_{j=1}^{\alpha_1} A_j^{(1)}$,
- (2) $A_J^{(2)} = \{i, i+1\}$ for some J ,
- (3) $A_K^{(3)} = \{i, i+1, j | j > i+1\}$ for some K .

We have

$$\begin{aligned}
& \mathcal{O}_n(z_1, \dots, z_i, z_{i+1}, \dots, z_n) - \mathcal{O}_n(z_1, \dots, z_{i+1}, z_i, \dots, z_n) \\
& = \tilde{\mathcal{O}}_n(z_1, \dots, z_i, z_{i+1}, \dots, z_n) - \tilde{\mathcal{O}}_n(z_1, \dots, z_{i+1}, z_i, \dots, z_n).
\end{aligned} \tag{3.40}$$

Here we have set

$$\begin{aligned}
& \tilde{\mathcal{O}}_n(z_1, z_2, z_3, \dots, z_n) = \sum_{\substack{\alpha_1, \alpha_2, \alpha_3 \geq 0 \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 = n}} (-c)^{\alpha_2 + 2\alpha_3} \Delta(x^3)^{\alpha_3} \\
& \times \left(\sum_{\substack{\{A_j^{(s)}\} \\ s=1,2,3 \\ j=1, \dots, \alpha_s \\ \{i, i+1\} \subset \cup_{j=1}^{\alpha_s} A_j^{(1)}}} + \sum_{\substack{\{A_J^{(s)}\} \\ s=1,2,3 \\ j=1, \dots, \alpha_s \\ A_J^{(2)} = \{i, i+1\} \text{ for some } J}} + \sum_{\substack{\{A_K^{(s)}\} \\ s=1,2,3 \\ j=1, \dots, \alpha_s \\ A_K^{(3)} = \{i, i+1, j | i+1 < j\} \text{ for some } K}} \right)
\end{aligned}$$

$$\begin{aligned}
& \times \prod_{\substack{j \in A_{Min}^{(1)} \\ \overrightarrow{j}}} T_1(z_j) \prod_{\substack{j \in A_{Min}^{(2)} \\ \overrightarrow{j}}} T_2(x^{-1}z_j) \prod_{\substack{j \in A_{Min}^{(3)} \\ \overrightarrow{j}}} T_3(z_j) \\
& \times \prod_{\substack{j=1 \\ j_1=A_{j,1}^{(2)} \\ j_2=A_{j,2}^{(2)}}}^{\alpha_2} \delta\left(\frac{x^2 z_{j_2}}{z_{j_1}}\right) \prod_{\substack{j=1 \\ j_1=A_{j,1}^{(3)} \\ j_2=A_{j,2}^{(3)} \\ j_3=A_{j,3}^{(3)}}}^{\alpha_3} \left(\delta\left(\frac{x^2 z_{j_2}}{z_{j_1}}\right) \delta\left(\frac{x^2 z_{j_1}}{z_{j_3}}\right) + \delta\left(\frac{x^2 z_{j_3}}{z_{j_1}}\right) \delta\left(\frac{x^2 z_{j_1}}{z_{j_2}}\right) \right) \\
& \times \prod_{t=1}^3 \prod_{\substack{1 \leq j < k \leq n \\ j, k \in A_{Min}^{(t)}}} g_{t,t} \left(\frac{z_k}{z_j} \right) \prod_{\substack{j \in A_{Min}^{(1)} \\ k \in A_{Min}^{(2)}}} g_{1,2} \left(\frac{x^{-1} z_k}{z_j} \right) \prod_{\substack{j \in A_{Min}^{(1)} \\ k \in A_{Min}^{(3)}}} g_{1,3} \left(\frac{z_k}{z_j} \right) \prod_{\substack{j \in A_{Min}^{(2)} \\ k \in A_{Min}^{(3)}}} g_{2,3} \left(\frac{x z_k}{z_j} \right). \tag{3.41}
\end{aligned}$$

Let us consider the formulae relating to the first summation in $\tilde{\mathcal{O}}_n(z_1, \dots, z_i, z_{i+1}, \dots, z_n) - \tilde{\mathcal{O}}_n(z_1, \dots, z_{i+1}, z_i, \dots, z_n)$. Let us start from

$$\begin{aligned}
& \sum_{\substack{\alpha_1, \alpha_2, \alpha_3 \geq 0 \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 = n}} (-c)^{\alpha_2 + 2\alpha_3} \Delta(x^3)^{\alpha_3} \sum_{\substack{\{A_j^{(s)}\}_{j=1, \dots, \alpha_s} \\ s=1, 2, 3 \\ \{i, i+1\} \subset \cup_{j=1}^{\alpha_s} A_j^{(1)}}} \\
& \times \prod_{\substack{j \in A_{Min}^{(1)} \\ j < i}} T_1(z_j) (g_{11}(z_{i+1}/z_i) T_1(z_i) T_1(z_{i+1}) - g_{11}(z_{i+1}/z_i) T_1(z_{i+1}) T_1(z_i)) \prod_{\substack{j \in A_{Min}^{(1)} \\ i+1 < j}} T_1(z_j) \\
& \times \prod_{\substack{j \in A_{Min}^{(2)} \\ \overrightarrow{j}}} T_2(x^{-1}z_j) \prod_{\substack{j \in A_{Min}^{(3)} \\ \overrightarrow{j}}} T_3(z_j) \\
& \times \prod_{\substack{j=1 \\ j_1=A_{j,1}^{(2)} \\ j_2=A_{j,2}^{(2)}}}^{\alpha_2} \delta\left(\frac{x^2 z_{j_2}}{z_{j_1}}\right) \prod_{\substack{j=1 \\ j_1=A_{j,1}^{(3)} \\ j_2=A_{j,2}^{(3)} \\ j_3=A_{j,3}^{(3)}}}^{\alpha_3} \left(\delta\left(\frac{x^2 z_{j_2}}{z_{j_1}}\right) \delta\left(\frac{x^2 z_{j_1}}{z_{j_3}}\right) + \delta\left(\frac{x^2 z_{j_3}}{z_{j_1}}\right) \delta\left(\frac{x^2 z_{j_1}}{z_{j_2}}\right) \right) \\
& \times \prod_{t=1}^3 \prod_{\substack{1 \leq j < k \leq n \\ j, k \in A_{Min}^{(t)}}} g_{t,t} \left(\frac{z_k}{z_j} \right) \prod_{\substack{j \in A_{Min}^{(1)} \\ k \in A_{Min}^{(2)}}} g_{1,2} \left(\frac{x^{-1} z_k}{z_j} \right) \prod_{\substack{j \in A_{Min}^{(1)} \\ k \in A_{Min}^{(3)}}} g_{1,3} \left(\frac{z_k}{z_j} \right) \prod_{\substack{j \in A_{Min}^{(2)} \\ k \in A_{Min}^{(3)}}} g_{2,3} \left(\frac{x z_k}{z_j} \right). \tag{3.42}
\end{aligned}$$

By applying the weakly sense relation in Proposition 3.3, let us change the ordering of $g_{11}(z_{i+1}/z_i) T_1(z_i) T_1(z_{i+1}) - g_{11}(z_{i+1}/z_i) T_1(z_{i+1}) T_1(z_i)$ and $T_1(z_j)$ for $j > i+1$ ($j \in A_{Min}^{(1)}$). We have

$$\sum_{\substack{\alpha_1, \alpha_2, \alpha_3 \geq 0 \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 = n}} (-c)^{\alpha_2 + 2\alpha_3} \Delta(x^3)^{\alpha_3} \sum_{\substack{\{A_j^{(s)}\}_{j=1, \dots, \alpha_s} \\ s=1, 2, 3 \\ \{i, i+1\} \subset \cup_{j=1}^{\alpha_s} A_j^{(1)}}}$$

$$\begin{aligned}
& \times \prod_{\substack{\overrightarrow{j \in A_{Min}^{(1)}} \\ j \in A_{Min}^{(1)} - \{i, i+1\}}} T_1(z_j) T_2(x^{-1}z_i) \prod_{\substack{\overrightarrow{j \in A_{Min}^{(2)}} \\ j \in A_{Min}^{(2)}}} T_2(x^{-1}z_j) \prod_{\substack{\overrightarrow{j \in A_{Min}^{(3)}} \\ j \in A_{Min}^{(3)}}} T_3(z_j) \\
& \times \delta\left(\frac{x^2 z_{i+1}}{z_i}\right) \prod_{\substack{j=1 \\ j_1=A_{j,1}^{(2)} \\ j_2=A_{j,2}^{(2)}}}^{\alpha_2} \delta\left(\frac{x^2 z_{j_2}}{z_{j_1}}\right) \prod_{\substack{j=1 \\ j_1=A_{j,1}^{(3)} \\ j_2=A_{j,2}^{(3)} \\ j_3=A_{j,3}^{(3)}}}^{\alpha_3} \left(\delta\left(\frac{x^2 z_{j_2}}{z_{j_1}}\right) \delta\left(\frac{x^2 z_{j_1}}{z_{j_3}}\right) + \delta\left(\frac{x^2 z_{j_3}}{z_{j_1}}\right) \delta\left(\frac{x^2 z_{j_1}}{z_{j_2}}\right) \right) \\
& \times \prod_{\substack{1 \leq j < k \leq n \\ j, k \in A_{Min}^{(1)} - \{i, i+1\}}} g_{1,1}\left(\frac{z_k}{z_j}\right) \prod_{\substack{1 \leq j < k \leq n \\ j, k \in A_{Min}^{(2)}}} g_{2,2}\left(\frac{z_k}{z_j}\right) \prod_{j \in A_{Min}^{(2)}} g_{2,2}\left(\frac{z_j}{z_i}\right) \prod_{\substack{1 \leq j < k \leq n \\ j, k \in A_{Min}^{(3)}}} g_{3,3}\left(\frac{z_k}{z_j}\right) \\
& \times \prod_{\substack{j \in A_{Min}^{(1)} - \{i, i+1\} \\ k \in A_{Min}^{(2)} \cup \{i\}}} g_{1,2}\left(\frac{x^{-1}z_k}{z_j}\right) \prod_{\substack{j \in A_{Min}^{(1)} - \{i, i+1\} \\ k \in A_{Min}^{(3)}}} g_{1,3}\left(\frac{z_k}{z_j}\right) \prod_{\substack{j \in A_{Min}^{(2)} \cup \{i\} \\ k \in A_{Min}^{(3)}}} g_{2,3}\left(\frac{xz_k}{z_j}\right) \\
& - \sum_{\substack{\alpha_1, \alpha_2, \alpha_3 \geq 0 \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 = n}} \sum_{\substack{\{A_j^{(s)}\} \\ s=1,2,3 \\ j=1, \dots, \alpha_s \\ \{i, i+1\} \subset \cup_{j=1}^{\alpha_s} A_j^{(1)}}} \sum_{\substack{j \in A_{Min}^{(1)} \\ i+1 < j}} (-c)^{\alpha_2 + 2(\alpha_3 + 1)} \Delta(x^3)^{\alpha_3 + 1} \\
& \times \prod_{\substack{\overrightarrow{j \in A_{Min}^{(1)}} \\ j \in A_{Min}^{(1)} - \{i, i+1\}}} T_1(z_j) T_3(z_i) \left(\delta\left(\frac{x^2 z_{i+1}}{z_i}\right) \delta\left(\frac{x^2 z_{z_i}}{z_j}\right) + \delta\left(\frac{x^2 z_{j_j}}{z_i}\right) \delta\left(\frac{x^2 z_i}{z_{i+1}}\right) \right) \\
& \times \prod_{\substack{\overrightarrow{j \in A_{Min}^{(2)}} \\ j \in A_{Min}^{(2)}}} T_2(x^{-1}z_j) \prod_{\substack{\overrightarrow{j \in A_{Min}^{(3)}} \\ j \in A_{Min}^{(3)}}} T_3(z_j) \\
& \times \prod_{\substack{j=1 \\ j_1=A_{j,1}^{(2)} \\ j_2=A_{j,2}^{(2)}}}^{\alpha_2} \delta\left(\frac{x^2 z_{j_2}}{z_{j_1}}\right) \prod_{\substack{j=1 \\ j_1=A_{j,1}^{(3)} \\ j_2=A_{j,2}^{(3)} \\ j_3=A_{j,3}^{(3)}}}^{\alpha_3} \left(\delta\left(\frac{x^2 z_{j_2}}{z_{j_1}}\right) \delta\left(\frac{x^2 z_{j_1}}{z_{j_3}}\right) + \delta\left(\frac{x^2 z_{j_3}}{z_{j_1}}\right) \delta\left(\frac{x^2 z_{j_1}}{z_{j_2}}\right) \right) \\
& \times \prod_{\substack{1 \leq j < k \leq n \\ j, k \in A_{Min}^{(1)} - \{i, i+1\}}} g_{1,1}\left(\frac{z_k}{z_j}\right) \prod_{\substack{1 \leq j < k \leq n \\ j, k \in A_{Min}^{(2)}}} g_{2,2}\left(\frac{z_k}{z_j}\right) \prod_{\substack{1 \leq j < k \leq n \\ j, k \in A_{Min}^{(3)}}} g_{3,3}\left(\frac{z_k}{z_j}\right) \prod_{j \in A_{Min}^{(3)}} g_{3,3}\left(\frac{z_j}{z_i}\right) \\
& \times \prod_{\substack{j \in A_{Min}^{(1)} - \{i, i+1\} \\ k \in A_{Min}^{(2)}}} g_{1,2}\left(\frac{x^{-1}z_k}{z_j}\right) \prod_{\substack{j \in A_{Min}^{(1)} - \{i, i+1\} \\ k \in A_{Min}^{(3)} \cup \{i\}}} g_{1,3}\left(\frac{z_k}{z_j}\right) \prod_{\substack{j \in A_{Min}^{(2)} \\ k \in A_{Min}^{(3)} \cup \{i\}}} g_{2,3}\left(\frac{xz_k}{z_j}\right) \prod_{j \in A_{Min}^{(2)}} g_{3,2}\left(\frac{z_j}{z_i}\right) \\
& - (z_i \leftrightarrow z_{i+1}).
\end{aligned} \tag{3.43}$$

By using the weakly sense relations in Proposition 3.4, we move the operators $T_2(z), T_3(z)$ to the right. By changing variables $\{A_j^{(s)}\}$ to $\{B_j^{(s)}\}$, we have

$$\begin{aligned}
& - \sum_{\substack{\beta_1, \beta_2, \beta_3 \geq 0 \\ \beta_1 + 2\beta_2 + 3\beta_3 = n}} (-c)^{\beta_2 + 2\beta_3} \Delta(x^3)^{\beta_3} \left(\sum_{\substack{\{B_j^{(s)}\} \\ j=1, \dots, \beta_s \\ B_J^{(2)} = \{i, i+1\} \text{ for some } J}} + \sum_{\substack{\{B_j^{(s)}\} \\ j=1, \dots, \alpha_s \\ B_K^{(3)} = \{i, i+1, j | i+1 < j\} \text{ for some } K}} \right) \\
& \times \prod_{j \in \overrightarrow{B_{Min}^{(1)}}} T_1(z_j) \prod_{j \in \overrightarrow{B_{Min}^{(2)}}} T_2(x^{-1}z_j) \prod_{j \in \overrightarrow{B_{Min}^{(3)}}} T_3(z_j) \\
& \times \prod_{\substack{j=1 \\ j_1=B_{j,1}^{(2)} \\ j_2=B_{j,2}^{(2)}}}^{\beta_2} \delta\left(\frac{x^2 z_{j_2}}{z_{j_1}}\right) \prod_{\substack{j=1 \\ j_1=B_{j,1}^{(3)} \\ j_2=B_{j,2}^{(3)} \\ j_3=B_{j,3}^{(3)}}}^{\beta_3} \left(\delta\left(\frac{x^2 z_{j_2}}{z_{j_1}}\right) \delta\left(\frac{x^2 z_{j_1}}{z_{j_3}}\right) + \delta\left(\frac{x^2 z_{j_3}}{z_{j_1}}\right) \delta\left(\frac{x^2 z_{j_1}}{z_{j_2}}\right) \right) \\
& \times \prod_{s=1}^3 \prod_{\substack{1 \leq j < k \leq n \\ j, k \in B_{Min}^{(s)}}} g_{s,s} \left(\frac{z_k}{z_j} \right) \prod_{\substack{j \in B_{Min}^{(1)} \\ k \in B_{Min}^{(2)}}} g_{1,2} \left(\frac{x^{-1} z_k}{z_j} \right) \prod_{\substack{j \in B_{Min}^{(1)} \\ k \in B_{Min}^{(3)}}} g_{1,3} \left(\frac{z_k}{z_j} \right) \prod_{\substack{j \in B_{Min}^{(2)} \\ k \in B_{Min}^{(3)}}} g_{2,3} \left(\frac{x z_k}{z_j} \right) \\
& - (z_i \leftrightarrow z_{i+1}). \tag{3.44}
\end{aligned}$$

This is exactly the same as the second and the third summation up to signature. Now we have shown S_n -invriance of \mathcal{O}_n in “the weak sense”.

For the second we summareize the proof for \widehat{sl}_N . Formulae are more complicated , however the idea of the proof is the same. In order to show S_n -invariance, it is enough to show the case of the permutations $\sigma = (i, i+1)$ for $1 \leq i \leq n-1$. Because of the cancellations, the difference $\mathcal{O}_n(\dots, z_i, z_{i+1}, \dots) - \mathcal{O}_n(\dots, z_{i+1}, z_i, \dots)$ has simplification. We don't have to consider every summation $\sum_{\substack{\{A_j^{(s)}\} \\ s=1, \dots, N \\ j=1, \dots, \alpha_s}}$ in the definition of \mathcal{O}_n . We only have to consider the summation of the following N -cases for $\sigma = (i, i+1)$

- (1) $\{i, i+1\} \subset \cup_{j=1}^{\alpha_1} A_j^{(1)}$,
- (2) $A_J^{(2)} = \{i, i+1\}$ for some J ,
- (3) $A_J^{(3)} = \{i, i+1, j_3 | i+1 < j_3\}$ for some J ,
-
- (s) $A_J^{(s)} = \{i, i+1, j_3, \dots, j_s | i+1 < j_3 < \dots < j_s\}$ for some J ,
-

$$(N) A_J^{(N)} = \{i, i+1, j_3, j_4, \dots, j_N | i+1 < j_3 < j_4 < \dots < j_N\} \text{ for some } J.$$

We have

$$\begin{aligned} & \mathcal{O}_n(z_1, \dots, z_i, z_{i+1}, \dots, z_n) - \mathcal{O}_n(z_1, \dots, z_{i+1}, z_i, \dots, z_n) \\ &= \tilde{\mathcal{O}}_n(z_1, \dots, z_i, z_{i+1}, \dots, z_n) - \tilde{\mathcal{O}}_n(z_1, \dots, z_{i+1}, z_i, \dots, z_n). \end{aligned} \quad (3.45)$$

Here we have set

$$\begin{aligned} \tilde{\mathcal{O}}_n(z_1, \dots, z_i, z_{i+1}, \dots, z_n) &= \sum_{\substack{\alpha_1, \alpha_2, \dots, \alpha_N \geq 0 \\ \alpha_1 + 2\alpha_2 + \dots + N\alpha_N = n}} \prod_{t=1}^N \left((-c)^{t-1} \prod_{u=1}^{t-2} \Delta(x^{2u+1})^{t-u-1} \right)^{\alpha_t} \\ &\times \left(\sum_{\substack{\{A_j^{(s)}\}_{s=1, \dots, N} \\ j=1, \dots, \alpha_s \\ \{i, i+1\} \subset \cup_{j=1}^{\alpha_s} A_j^{(1)}}} + \sum_{t=2}^N \sum_{\substack{\{A_j^{(s)}\}_{s=1, \dots, N} \\ j=1, \dots, \alpha_s \\ A_J^{(t)} = \{i, i+1, j_3, \dots, j_t | i+1 < j_3 < \dots < j_t\} \text{ for some } J}} \right) \prod_{1 \leq s \leq N} \prod_{j \in \overrightarrow{A_{Min}^{(s)}}} T_s(x^{-1+s+\lceil \frac{s}{2} \rceil} z_j) \\ &\times \prod_{t=1}^N \prod_{j=1}^{\alpha_t} \sum_{\substack{\sigma \in S_t \\ \sigma(1)=1 \\ j_1 = A_{j,1}^{(t)} \\ \dots \\ j_t = A_{j,t}^{(t)}}} \prod_{u=1}^t \delta \left(\frac{x^2 z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}} \right) \\ &\times \prod_{t=1}^N \prod_{\substack{1 \leq j < k \leq n \\ j, k \in \overrightarrow{A_{Min}^{(t)}}}} g_{t,t} \left(\frac{z_k}{z_j} \right) \prod_{1 \leq t < u \leq N} \prod_{\substack{j \in \overrightarrow{A_{Min}^{(t)}} \\ k \in \overrightarrow{A_{Min}^{(u)}}}} g_{t,u} \left(x^{u-t-2\lceil \frac{u-t}{2} \rceil} \frac{z_k}{z_j} \right). \end{aligned} \quad (3.46)$$

Let us consider the formulae relating to the first term in $\tilde{\mathcal{O}}_n(z_1, \dots, z_i, z_{i+1}, \dots, z_n) - \tilde{\mathcal{O}}_n(z_1, \dots, z_{i+1}, z_i, \dots, z_n)$. Let us start from

$$\begin{aligned} & \sum_{\substack{\alpha_1 \geq 2 \text{ and } \alpha_2, \dots, \alpha_N \geq 0 \\ \alpha_1 + 2\alpha_2 + \dots + N\alpha_N = n}} \prod_{t=1}^N \left((-c)^{t-1} \prod_{u=1}^{t-2} \Delta(x^{2u+1})^{t-u-1} \right)^{\alpha_t} \sum_{\substack{\{A_j^{(s)}\}_{s=1, \dots, N} \\ j=1, \dots, \alpha_s \\ \{i, i+1\} \subset \cup_{j=1}^{\alpha_s} A_j^{(1)}}} \\ & \times \prod_{\substack{j \in \overrightarrow{A_{Min}^{(1)}} \\ j < i}} T_1(z_j) \cdot T_1(z_i) T_1(z_{i+1}) \cdot \prod_{\substack{j \in \overrightarrow{A_{Min}^{(1)}} \\ i+1 < j}} T_1(z_j) \prod_{2 \leq s \leq N} \prod_{j \in \overrightarrow{A_{Min}^{(s)}}} T_s(x^{-1+s+\lceil \frac{s}{2} \rceil} z_j) \\ & \times \prod_{t=2}^N \prod_{j=1}^{\alpha_t} \sum_{\substack{\sigma \in S_t \\ \sigma(1)=1 \\ j_1 = A_{j,1}^{(t)} \\ \dots \\ j_t = A_{j,t}^{(t)}}} \prod_{u=1}^t \delta \left(\frac{x^2 z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}} \right) \end{aligned}$$

$$\times \prod_{t=1}^N \prod_{\substack{1 \leq j < k \leq n \\ j, k \in A_{Min}^{(t)}}} g_{t,t} \left(\frac{z_k}{z_j} \right) \prod_{1 \leq t < u \leq N} \prod_{\substack{j \in A_{Min}^{(t)} \\ k \in A_{Min}^{(u)}}} g_{t,u} \left(x^{u-t-2[\frac{u-t}{2}]} \frac{z_k}{z_j} \right) - (z_i \leftrightarrow z_{i+1}). \quad (3.47)$$

By using the weakly sense relations in Proposition ?? we change the ordering of $T_1(z_i)T_1(z_{i+1})$

and $\prod_{\substack{j \in A_{Min}^{(1)} \\ i+1 < j}} T_1(z_j)$. We have

$$\begin{aligned} & - \sum_{\substack{\alpha_1+2\alpha_2+\dots+N\alpha_N=n \\ \alpha_1 \geq 2 \text{ and } \alpha_2, \dots, \alpha_N \geq 0}} \sum_{t=2}^N \sum_{\substack{\{A_j^{(s)}\} \\ \{i, i+1\} \subset A_{Min}^{(1)}}} \sum_{\substack{i+1 < j_3 < \dots < j_t \\ j_3, \dots, j_t \in A_{Min}^{(1)}}} \\ & \times \prod_{\substack{s=1 \\ s \neq t}}^N \left((-c)^{s-1} \prod_{u=1}^{s-2} \Delta(x^{2u+1})^{s-u-1} \right)^{\alpha_s} \left((-c)^{t-1} \prod_{u=1}^{t-2} \Delta(x^{2u+1})^{t-u-1} \right)^{\alpha_t+1} \\ & \times \prod_{j \in A_{Min}^{(1)} - \{i, i+1\}} T_1(z_j) T_t(x^{-1+t-2[\frac{t}{2}]} z_i) \prod_{2 \leq s \leq N} \prod_{j \in A_{Min}^{(s)}} T_s(x^{-1+s-2[\frac{s}{2}]} z_j) \\ & \times \sum_{\substack{\sigma \in S_t \\ \sigma(1)=1}} \prod_{\substack{u=1 \\ u \neq [\frac{t}{2}]+1 \\ j_1=1, j_2=2}}^t \delta \left(\frac{x^2 z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}} \right) \prod_{s=2}^N \prod_{j_1=A_{j,1}^{(s)}, \dots, j_s=A_{j,s}^{(s)}}^{\alpha_s} \sum_{\substack{\sigma \in S_s \\ \sigma(1)=1}} \prod_{\substack{u=1 \\ u \neq [\frac{s}{2}]+1}}^s \delta \left(\frac{x^2 z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}} \right) \\ & \times \prod_{\substack{j < k \\ j, k \in A_{Min}^{(1)} - \{i, i+1\}}} g_{1,1}(z_k/z_j) \prod_{s=2}^N \prod_{\substack{j < k \\ j, k \in A_{Min}^{(s)}}} g_{s,s}(z_k/z_j) \prod_{2 \leq t < u \leq N} \prod_{\substack{j \in A_{Min}^{(t)} \\ k \in A_{Min}^{(u)}}} g_{t,u} \left(x^{u-t-2[\frac{u-t}{2}]} z_k/z_j \right) \\ & \times \prod_{s=2}^N \prod_{\substack{j \in A_{Min}^{(1)} - \{i, i+1\} \\ k \in A_{Min}^{(s)}}} g_{1,s} \left(x^{-1+s-2[\frac{s}{2}]} \frac{z_k}{z_j} \right) \prod_{j \in A_{Min}^{(1)} - \{i, i+1\}} g_{1,t} \left(x^{-1+t-2[\frac{t}{2}]} \frac{z_i}{z_j} \right) \\ & \times \prod_{s=2}^N \prod_{j \in A_{Min}^{(s)}} g_{t,s} \left(x^{s-t-2[\frac{s-t}{2}]} \frac{z_j}{z_i} \right) - (z_i \leftrightarrow z_{i+1}). \end{aligned} \quad (3.48)$$

We change the summation variables $\{A_j^{(s)}\}$ in

$$\sum_{0 \leq t \leq N-2} \sum_{\substack{\alpha_1+2\alpha_2+\dots+N\alpha_N=n \\ \alpha_1 \geq 2 \text{ and } \alpha_2, \dots, \alpha_N \geq 0}} \sum_{\substack{\{A_j^{(s)}\} \\ j=1, \dots, N \\ \{i, i+1\} \subset A_{Min}^{(1)}}} \sum_{\substack{s=1, \dots, N \\ j_3, \dots, j_{t+2} \in A_{Min}^{(1)} \\ i+1 < j_3 < \dots < j_{t+2}}} \quad (3.49)$$

to the following $\{B_j^{(s)}\}$,

$$\sum_{0 \leq t \leq N-2} \sum_{\substack{\beta_1+2\beta_2+\dots+N\beta_N=n \\ \beta_1, \beta_2, \dots, \beta_N \geq 0}} \sum_{\substack{\{B_j^{(s)}\} \\ s=1, \dots, N \\ j=1, \dots, \beta_s \\ B_J^{(t)} = \{i, i+1, j_3, \dots, j_{t+2} | i+1 < j_3 < \dots < j_{t+2}\} \text{ for some } J}} \quad (3.50)$$

Simultaneously let us use the weakly sense relations in Proposition 3.4 on the commutation relations between $T_i(z)$ and $T_j(w)$ for $i, j \geq 2$ and make the ordering

$$\prod_{\overrightarrow{1 \leq s \leq N}} \prod_{j \in \overrightarrow{B_{Min}^{(s)}}} T_j(x^{-1+s-2[\frac{s}{2}]} z_j),$$

where $B_{Min}^{(s)} = \{Min(B_1^{(s)}), \dots, Min(B_{\alpha_s}^{(s)})\}$. We have exactly the same summation of the second to N -th terms of $\tilde{\mathcal{O}}_n(\dots, z_i, z_{i+1}, \dots) - \tilde{\mathcal{O}}_n(\dots, z_{i+1}, z_i, \dots)$ up to signature. Now we have shown theorem for \widehat{sl}_N case. Q.E.D.

3.5 Derivation of Laurent-Series Formulae

In this section we give proof of Theorem 3.2.

Proof of Theorem 3.2 At first we give proof for \widehat{sl}_3 case. We start from the integral representation \mathcal{I}_n in (3.2). Let us pay attention to the poles $z_{J_1} = x^{-2}z_1, (2 \leq J_1 \leq n)$. We have

$$\begin{aligned} \mathcal{I}_n &= \int \cdots \int_{C(1)} \prod_{j=1}^n \frac{dz_j}{2\pi\sqrt{-1}z_j} \prod_{1 \leq j < k \leq n} h(u_k - u_j) \prod_{\overrightarrow{1 \leq j \leq n}} T_1(z_j) \\ &- \sum_{J_1=2}^n \int \cdots \int_{\widehat{C}(J_1)} \prod_{\substack{j=1 \\ j \neq J_1}}^n \frac{dz_j}{2\pi\sqrt{-1}z_j} \int_{C_{x^{-2}z_1}} \frac{dz_{J_1}}{2\pi\sqrt{-1}z_{J_1}} \prod_{1 \leq j < k \leq n} h(u_k - u_j) \prod_{\overrightarrow{1 \leq j \leq n}} T_1(z_j). \end{aligned} \quad (3.51)$$

Here we have set

$$\begin{aligned} C(1) &: |x^{-2}z_k| < |z_1| < |x^{2-2s}z_k|, \quad (2 \leq k \leq n) \\ &|x^{-2}z_k| < |z_j| < |x^2z_k|, \quad (2 \leq j < k \leq n), \end{aligned} \quad (3.52)$$

$$\begin{aligned} \widehat{C}(J_1) &: |x^{-2}z_k| < |z_1| < |x^{2-2s}z_k|, \quad (2 \leq k \leq J_1 - 1) \\ &|x^{-2}z_k| < |z_1| < |x^2z_k|, \quad (J_1 + 1 \leq k \leq n), \\ &|x^{-2}z_k| < |z_j| < |x^2z_k|, \quad (2 \leq j < k \leq n; j, k \neq J_1). \end{aligned} \quad (3.53)$$

Here $C_{x^{-2}z_1}$ is a small circle which encircle $x^{-2}z_1$ anticlockwise. The region $\{(z_1, z_k) \in \mathbb{C}^2 | |x^{-2}z_k| < |z_1| < |x^{2-2s}z_k|\}$ for $2 \leq k \leq J_1$, are annulus. Hence the defining relations of the deformed W -algebra can be used. Let us change the ordering of $T_1(z_1)$ and $T_1(z_k)$

for $2 \leq k \leq J_1 - 1$, and take the residue of $T_1(z_1)T_1(z_{J_1})$ at $z_{J_1} = x^{-2}z_1$. We have

$$\begin{aligned}
& \int \cdots \int_{\widehat{C}(J_1)} \prod_{\substack{j=1 \\ j \neq J_1}}^n \frac{dz_j}{2\pi\sqrt{-1}z_j} \int_{C_{x^{-2}z_1}} \frac{dz_{J_1}}{2\pi\sqrt{-1}z_{J_1}} \prod_{1 \leq j < k \leq n} h(u_k - u_j) \prod_{1 \leq j \leq n} T_1(z_j) \\
&= c \int \cdots \int_{C(J_1)} \prod_{\substack{j=1 \\ j \neq J_1}}^n \frac{dz_j}{2\pi\sqrt{-1}z_j} \prod_{2 \leq j \leq J_1-1} T_1(z_j) \cdot T_2(x^{-1}z_1) \cdot \prod_{J_1+1 \leq j \leq n} T_1(z_j) \\
&\times \prod_{\substack{2 \leq j < k \leq n \\ j, k \neq J_1}} h_{11}(u_k - u_j) \prod_{j=2}^{J_1-1} h_{12} \left(u_1 - u_j - \frac{1}{2} \right) \prod_{j=J_1+1}^n h_{21} \left(u_j - u_1 + \frac{1}{2} \right). \quad (3.54)
\end{aligned}$$

Here we have set

$$\begin{aligned}
C(J_1) \quad : \quad & |x^{-2}z_j| < |z_1| < |x^4z_j|, \quad (2 \leq j \leq n; j \neq J_1), \\
& |x^{-2}z_k| < |z_j| < |x^2z_k|, \quad (2 \leq j < k \leq n; j, k \neq J_1). \quad (3.55)
\end{aligned}$$

Let us pay attention to the poles at $z_{J_2} = x^2z_1$, ($2 \leq J_2 \leq n; J_2 \neq J_1$). We deform the RHS of (3.54) to the following.

$$\begin{aligned}
& c \int \cdots \int_{C(J_1)(J_1)} \prod_{\substack{j=1 \\ j \neq J_1}}^n \frac{dz_j}{2\pi\sqrt{-1}z_j} \prod_{2 \leq j \leq J_1-1} T_1(z_j) \cdot T_2(x^{-1}z_1) \cdot \prod_{J_1+1 \leq j \leq n} T_1(z_j) \\
&\times \prod_{\substack{2 \leq j < k \leq n \\ j, k \neq J_1}} h_{11}(u_k - u_j) \prod_{j=2}^{J_1-1} h_{12} \left(u_1 - u_j - \frac{1}{2} \right) \prod_{j=J_1+1}^n h_{21} \left(u_j - u_1 + \frac{1}{2} \right) \\
&- c \sum_{\substack{J_2=2 \\ J_2 \neq J_1}}^n \int \cdots \int_{\widehat{C}(J_1)(J_2)} \prod_{\substack{j=1 \\ j \neq J_1, J_2}}^n \frac{dz_j}{2\pi\sqrt{-1}z_j} \int_{C_{x^2z_1}} \frac{dz_{J_2}}{2\pi\sqrt{-1}z_{J_2}} \\
&\times \prod_{2 \leq j \leq J_1-1} T_1(z_j) \cdot T_2(x^{-1}z_1) \cdot \prod_{J_1+1 \leq j \leq n} T_1(z_j) \\
&\times \prod_{\substack{2 \leq j < k \leq n \\ j, k \neq J_1}} h_{11}(u_k - u_j) \prod_{j=2}^{J_1-1} h_{12} \left(u_1 - u_j - \frac{1}{2} \right) \prod_{j=J_1+1}^n h_{21} \left(u_j - u_1 + \frac{1}{2} \right). \quad (3.56)
\end{aligned}$$

Here we have set

$$\begin{aligned}
C(J_1)(J_1) \quad : \quad & |x^{-2+2s}z_j| < |z_1| < |x^4z_j|, \quad (2 \leq j \leq n; j \neq J_1), \\
& |x^{-2}z_k| < |z_j| < |x^2z_k|, \quad (2 \leq j < k \leq n; j, k \neq J_1). \quad (3.57)
\end{aligned}$$

For $2 \leq J_2 < J_1 \leq n$ we set

$$\widehat{C}(J_1)(J_2) \quad : \quad |x^{-2+2s}z_j| < |z_1| < |x^4z_j|, \quad (J_2 \leq j \leq J_1 - 1),$$

$$\begin{aligned}
|x^{-2}z_j| &< |z_1| < |x^4z_j|, \quad (2 \leq j \leq J_2 - 1 \text{ or } J_1 + 1 \leq j \leq n), \\
|x^{-2}z_k| &< |z_j| < |x^2z_k|, \quad (2 \leq j < k \leq n; j, k \neq J_1).
\end{aligned} \tag{3.58}$$

For $2 \leq J_1 < J_2 \leq n$ we set

$$\begin{aligned}
\widehat{C}(J_1)(J_2) &: |x^{-2+2s}z_j| < |z_1| < |x^4z_j|, \quad (2 \leq j \leq J_2; j \neq J_1), \\
|x^{-2}z_j| &< |z_1| < |x^4z_j|, \quad (J_2 + 1 \leq j \leq n), \\
|x^{-2}z_k| &< |z_j| < |x^2z_k|, \quad (2 \leq j < k \leq n; j, k \neq J_1).
\end{aligned} \tag{3.59}$$

The above formulae for this integrand $C(J_1)(J_2)$ holds for $\text{Re}(s) > N \geq 3$. For $N = 2$ another treatment should be done. Let us study the first term $c \int \cdots \int_{C(J_1)(J_1)} \prod_{j \neq J_1} \frac{dz_j}{2\pi\sqrt{-1}} \frac{1}{z_j}$. See the integral contour $C(J_1)(J_1)$. The region $\{(z_1, z_j) \in \mathbb{C}^2 \mid |x^{-2+2s}z_j| < |z_1| < |x^4z_j|\}$ for $j \neq J_1$ are annulus. Hence the defining relations of the deformed W -algebra can be used. By using the weakly sense relation in Proposition 3.3, we deform the first term to the following.

$$\begin{aligned}
&c \int \cdots \int_{C(J_1)(J_1)} \prod_{\substack{j=1 \\ j \neq J_1}}^n \frac{dz_j}{2\pi\sqrt{-1}z_j} \prod_{\substack{2 \leq j \leq n \\ j \neq J_1}} T_1(z_j) \cdot T_2(x^{-1}z_1) \\
&\times \prod_{\substack{2 \leq j < k \leq n \\ j, k \neq J_1}} h_{11}(u_k - u_j) \prod_{\substack{j=2 \\ j \neq J_1}}^n h_{12} \left(u_1 - u_j - \frac{1}{2} \right).
\end{aligned} \tag{3.60}$$

Let us study the second term $-c \sum_{J_2 \neq 2, J_1} \int \cdots \int_{\widehat{C}(J_1)(J_2)} \prod_{j \neq J_1, J_2} \frac{dz_j}{2\pi\sqrt{-1}z_j} \int_{C_{x^2z_1}} \frac{dz_{J_2}}{2\pi\sqrt{-1}z_{J_2}}$. See the integral contour $\widehat{C}(J_1)(J_2)$. The region $\{(z_1, z_j) \in \mathbb{C}^2 \mid |x^{-2+2s}z_j| < |z_1| < |x^4z_j|\}$ for $2 \leq j \leq J_1$, are annulus for $\text{Re}(s) > N = 3$. Hence the defining relations of the deformed W -algebra can be used. Let us change the ordering of $T_1(z_{J_2})$ and $T_1(z_k)$ and make the product of the operators $T_1(z_{J_2})T_2(x^{-1}z_1)$ or $T_2(x^{-1}z_1)T_1(z_{J_2})$. Let us take the residue of $T_1(z_{J_2})T_2(x^{-1}z_1)$ and $T_2(x^{-1}z_1)T_1(z_{J_2})$ at $z_{J_2} = x^2z_1$ by regarding the weakly sense equation in Proposition 3.3. We have

$$\begin{aligned}
&c^2 \Delta(x^3) \sum_{\substack{J_2=2 \\ J_2 \neq J_1}}^n \int \cdots \int_{C(J_1)(J_2)} \prod_{\substack{j=1 \\ j \neq J_1, J_2}}^n \frac{dz_j}{2\pi\sqrt{-1}z_j} \prod_{\substack{2 \leq j \leq J_1-1 \\ j \neq J_2}} T_1(z_j) \cdot T_3(z_1) \cdot \prod_{\substack{J_1+1 \leq j \leq n \\ j \neq J_2}} T_1(z_j) \\
&\times \prod_{\substack{2 \leq j < k \leq n \\ j, k \neq J_1, J_2}} h_{11}(u_k - u_j) \prod_{\substack{j=2 \\ j \neq J_2}}^{J_1-1} h_{13}(u_1 - u_j) \prod_{\substack{j=J_1+1 \\ j \neq J_2}}^n h_{31}(u_j - u_1).
\end{aligned} \tag{3.61}$$

Here we have set

$$\begin{aligned} C(J_1)(J_2) : & \quad |x^{-4+2s}z_j| < |z_1| < |x^{4-2s}z_j|, \quad (2 \leq j \leq n; j \neq J_1, J_2), \\ & \quad |x^{-2}z_k| < |z_j| < |x^2z_k|, \quad (2 \leq j < k \leq n; j \neq J_1, J_2). \end{aligned} \quad (3.62)$$

This integral contour $C(J_1)(J_2)$ holds only for $N = 3$ case. For $N \geq 4$ case another treatment should be done. The region $\{(z_1, z_j) \in \mathbb{C}^2 \mid |x^{-4+2s}z_j| < |z_1| < |x^4z_j|\}$ are annulus. We move $T_3(z_j)$ to the right, and get

$$\begin{aligned} & c^2 \Delta(x^3) \sum_{\substack{J_2=2 \\ J_2 \neq J_1}}^n \int \cdots \int_{C(J_1)(J_2)} \prod_{\substack{j=1 \\ j \neq J_1, J_2}}^n \frac{dz_j}{2\pi\sqrt{-1}z_j} \prod_{\substack{2 \leq j \leq n \\ j \neq J_1, J_2}} T_1(z_j) \cdot T_3(z_1) \\ & \times \prod_{\substack{2 \leq j < k \leq n \\ j, k \neq J_1, J_2}} h_{11}(u_k - u_j) \prod_{\substack{j=2 \\ j \neq J_1, J_2}}^n h_{13}(u_1 - u_j). \end{aligned} \quad (3.63)$$

Summing up every terms, we have

$$\begin{aligned} \mathcal{I}_n &= \left(\sum_{\substack{A^{(1)}=\{1\} \\ A^{(2)}=A^{(3)}=\phi}} -c \sum_{\substack{A^{(2)}=\{1,j\} \\ A^{(1)}=A^{(3)}=\phi}} + 2!c^2 \Delta(x^3) \sum_{\substack{A^{(3)}=\{1,j,k\} \\ A^{(1)}=A^{(2)}=\phi}} \right) \\ & \times \int \cdots \int_{C\{A^{(1)}, A^{(2)}, A^{(3)}, A_c\}} \prod_{j \in A_c \cup A_{Min}^{(1)} \cup A_{Min}^{(2)} \cup A_{Min}^{(3)}} \frac{dz_j}{2\pi\sqrt{-1}z_j} \\ & \times \prod_{j \in A_c \cup A_{Min}^{(1)}} T_1(z_j) \prod_{j \in A_{Min}^{(2)}} T_2(x^{-1}z_j) \prod_{j \in A_{Min}^{(3)}} T_3(z_j) \prod_{\substack{j < k \\ j, k \in A_c \cup A_{Min}^{(1)}}} h_{11}(u_k - u_j) \\ & \times \prod_{k \in A_{Min}^{(2)}} \prod_{j \in A_{Min}^{(1)} \cup A_c} h_{12}\left(u_k - u_j - \frac{1}{2}\right) \prod_{k \in A_{Min}^{(3)}} \prod_{j \in A_{Min}^{(1)} \cup A_c} h_{13}(u_k - u_j). \end{aligned} \quad (3.64)$$

Here we set $A_c = \{1, 2, \dots, n\} - A^{(1)} \cup A^{(2)} \cup A^{(3)}$. We have set $A_{Min}^{(t)} = \{j_1\}$ for $A^{(t)} = \{j_1 < j_2 < \cdots < j_t\}$. Here we have set $C\{A^{(1)}, A^{(2)}, A^{(3)}, A_c\}$ by

$$\begin{aligned} & |x^{-2}z_k| < |z_1| < |x^{2-2s}z_k|, \quad (k \in A_c \text{ for } A^{(1)} \neq \phi), \\ & |x^{-2+2s}z_k| < |z_1| < |x^4z_k|, \quad (k \in A_c \text{ for } A^{(2)} \neq \phi), \\ & |x^{-4+2s}z_k| < |z_1| < |x^{4-2s}z_k|, \quad (k \in A_c \text{ for } A^{(3)} \neq \phi), \\ & |x^{-2}z_k| < |z_j| < |x^2z_k|, \quad (j < k; j, k \in A_c). \end{aligned} \quad (3.65)$$

Next we deform the part $\prod_{j \in A_c} T_1(z_j)$. Let us take the residue at $z_j = x^{-2}z_2$, and continue similar calculations as above. We use the weakly sense equations in Proposition 3.4 and

change the ordering of $T_2(x^{-1}w)$ and $T_3(w)$, without taking residues. Now we have shown theorem for \widehat{sl}_3 .

Now we begin Proof for \widehat{sl}_N case. Let us start from the integral representation \mathcal{I}_n . Proof for generl \widehat{sl}_N case is similar as those for \widehat{sl}_3 case. However it is not exactly the same. For example the integral contour $C(J_1)(J_2)$ of the equation (3.62) should be changed for $N \geq 4$ to the following.

$$\begin{aligned} |x^{-2}z_k| &< |z_1| < |x^{2-2s}z_k|, \quad (k \in A_c \text{ for } A^{(1)} \neq \phi), \\ |x^{-2+2s}z_k| &< |z_1| < |x^4z_k|, \quad (k \in A_c \text{ for } A^{(2)} \neq \phi), \\ |x^{-4}z_k| &< |z_1| < |x^{4-2s}z_k|, \quad (k \in A_c \text{ for } A^{(3)} \neq \phi), \\ |x^{-2}z_k| &< |z_j| < |x^2z_k|, \quad (j < k; j, k \in A_c). \end{aligned} \quad (3.66)$$

Here we have to take the residue at $z_k = x^{-4}z_1$ for $k \in A_c$. For proof for \widehat{sl}_N case, we have to take the residue deeper. Taking the residue relating to variable z_1 deeper, we have

$$\begin{aligned} \mathcal{I}_n &= \int \cdots \int_{C\{A_{Min}^{(1)}=\{1\}, A_{Min}^{(2)}=\phi, \dots, A_{Min}^{(N)}=\phi, A_c=\{2, \dots, n\}\}} \prod_{1 \leq j < k \leq n} h(u_k - u_j) \prod_{\overrightarrow{1 \leq j \leq n}} T_1(z_j) \\ &+ \sum_{k=1}^{Min(N, n)} (-c)^{k-1} (k-1)! \prod_{u=1}^{k-1} \Delta(x^{2u+1})^{k-u-1} \sum_{\substack{A^{(k)}=\{j_1=1, j_2, \dots, j_k\} \\ A^{(s)}=\phi, \quad (s \neq k)}} \\ &\times \int \cdots \int_{C\{A_{Min}^{(1)}, \dots, A_{Min}^{(N)}, A_c\}} \prod_{j \in A_{Min}^{(1)} \cup \dots \cup A_{Min}^{(N)} \cup A_c} \frac{dz_j}{2\pi\sqrt{-1}z_j} \\ &\times \prod_{j \in \overrightarrow{A_{Min}^{(1)} \cup A_c}} T_1(z_j) \prod_{j \in \overrightarrow{A_{Min}^{(2)}}} T_2(x^{-1}z_j) \cdots \prod_{j \in \overrightarrow{A_{Min}^{(N)}}} T_N\left(x^{-1+N-2[\frac{N}{2}]z_j}\right) \\ &\times \prod_{\substack{j < k \\ j, k \in \overrightarrow{A_{Min}^{(1)}}}} h_{11}(u_k - u_j) \prod_{t=2}^N \prod_{j \in A_{Min}^{(t)}} \prod_{k \in A_{Min}^{(1)} \cup A_c} h_{1,t}\left(u_j - u_k + \frac{t-1}{2} - [\frac{t}{2}]\right). \end{aligned} \quad (3.67)$$

Here we have set $C\{A_{Min}^{(1)}, A_{Min}^{(2)}, \dots, A_{Min}^{(N)}, A_c\}$ by

$$\begin{aligned} |x^{-2J-2}z_k| &< |z_1| < |x^{2J+2-2s}z_k|, \quad (k \in A_c; A^{(2J+1)} \neq \phi; J < \frac{N}{2} - 1), \\ |x^{-2J-2+2s}z_k| &< |z_1| < |x^{2J+2}z_k|, \quad (k \in A_c; A^{(2J)} \neq \phi; J < \frac{N}{2} - 3), \\ |x^{-N+2s}z_k| &< |z_1| < |x^{N-2s}z_k|, \quad (k \in A_c; A^{(N-1)} \neq \phi; N \text{ even}), \\ |x^{-N-1+2s}z_k| &< |z_1| < |x^{N+1-2s}z_k|, \quad (k \in A_c; A^{(N-1)} \neq \phi; N \text{ odd}), \\ |x^{-2}z_k| &< |z_j| < |x^2z_k|, \quad (j < k; j, k \in A_c). \end{aligned} \quad (3.68)$$

Next we deform the part $\prod_{j \in A_c} \rightarrow T_1(z_j)$. Let us take the residue at $z_J = x^{-2}z_2$, and continue similar calculations as above. We use the weakly sense equations in Propositions 3.3 and 3.4 and change the ordering of $T_i(z)$ and $T_j(w)$ for $i, j \geq 2$, without taking residues. We get Theorem for \widehat{sl}_N case. Calculations for \mathcal{I}_n^* are given by similar way. Q.E.D.

3.6 Proof of $[\mathcal{I}_m, \mathcal{I}_n] = 0$

In this section we show the commutation relation $[\mathcal{I}_m, \mathcal{I}_n] = 0$.

Proposition 3.7 *The following theta identity holds.*

$$\sum_{\sigma \in S_{m+n}} \prod_{j=1}^n \prod_{k=n+1}^{n+m} \frac{1}{h(u_{\sigma(k)} - u_{\sigma(j)})} = \sum_{\sigma \in S_{m+n}} \prod_{j=1}^m \prod_{k=m+1}^{n+m} \frac{1}{h(u_{\sigma(k)} - u_{\sigma(j)})}. \quad (3.69)$$

$$\sum_{\sigma \in S_{m+n}} \prod_{j=1}^n \prod_{k=n+1}^{n+m} \frac{1}{h^*(u_{\sigma(k)} - u_{\sigma(j)})} = \sum_{\sigma \in S_{m+n}} \prod_{j=1}^m \prod_{k=m+1}^{n+m} \frac{1}{h^*(u_{\sigma(k)} - u_{\sigma(j)})}. \quad (3.70)$$

Here $h(u)$ and $h^*(u)$ are given in (3.1).

This theta identity was written in [8] without proof. We have already summarized a proof of the theta identity in [13]. In order to make this paper self-contained, we re-summarize the proof, here.

Proof Let us set

$$\text{LHS}(n, m) = \sum_{\substack{J \subset \{1, 2, \dots, n+m\} \\ |J|=n}} \prod_{j \in J} \prod_{k \notin J} \frac{[u_k - u_j + 1]_s [u_k - u_j + r^*]_s}{[u_k - u_j]_s [u_k - u_j + r]_s}, \quad (3.71)$$

$$\text{RHS}(n, m) = \sum_{\substack{J^c \subset \{1, 2, \dots, n+m\} \\ |J^c|=m}} \prod_{j \in J^c} \prod_{k \notin J^c} \frac{[u_k - u_j + 1]_s [u_k - u_j + r^*]_s}{[u_k - u_j]_s [u_k - u_j + r]_s}. \quad (3.72)$$

We will show $\text{LHS}(n, m) = \text{RHS}(n, m)$. $\text{LHS}(n, m)$ and $\text{RHS}(n, m)$ are an elliptic functions. Therefore, from Liouville thorem, it is enough to check whether all the residues of $\text{LHS}(n, m)$ and $\text{RHS}(n, m)$ coincide or not. Candidates of poles are $u_\alpha = u_\beta$ ($\alpha \neq \beta$) and $u_\alpha = u_\beta - r$ ($\alpha \neq \beta$). Let us consider $u_\alpha = u_\beta$ ($\alpha \neq \beta$)

$$\begin{aligned} \text{LHS}(n, m) &= \left(\frac{[u_\alpha - u_\beta + 1]_s [u_\alpha - u_\beta + r^*]_s}{[u_\alpha - u_\beta]_s [u_\alpha - u_\beta - r]_s} + \frac{[u_\beta - u_\alpha + 1]_s [u_\beta - u_\alpha + r^*]_s}{[u_\beta - u_\alpha]_s [u_\beta - u_\alpha - r]_s} \right) \\ &\quad \times \sum_{\substack{J \cup J^c \subset \{1, \dots, n+m\} - \{\alpha, \beta\} \\ |J|=n-1}} \prod_{j \in J} \prod_{k \notin J} \frac{[u_k - u_j + 1]_s [u_k - u_j + r^*]_s}{[u_k - u_j]_s [u_k - u_j + r]_s} \end{aligned} \quad (3.73)$$

Hence we have $\text{Res}_{u_\alpha=u_\beta} \text{LHS}(n, m) = 0$. As the same manner we have $\text{Res}_{u_\alpha=u_\beta} \text{RHS}(n, m) = 0$. Therefore $u_\alpha = u_\beta$ is not pole. We only have to consider poles $u_\alpha = u_\beta$ ($\alpha \neq \beta$). We show the $\text{LHS}(n, m) = \text{RHS}(n, m)$ by the induction of the number $n + m$. We assume $n > m \geq 1$ without losing generality. At first we show the starting point $n > m = 1$.

$$\sum_{k=1}^{n+1} \prod_{\substack{j=1 \\ j \neq k}}^{n+1} \frac{[u_j - u_k + 1]_s [u_j - u_k + r^*]_s}{[u_j - u_k]_s [u_j - u_k + r]_s} = \sum_{k=1}^{n+1} \prod_{\substack{j=1 \\ j \neq k}}^{n+1} \frac{[u_k - u_j + 1]_s [u_k - u_j + r^*]_s}{[u_k - u_j]_s [u_k - u_j + r]_s} \quad (3.74)$$

Both $\text{LHS}(n, 1)$ and $\text{RHS}(n, 1)$ have simple poles at $u_\alpha = u_\beta - r$ ($\alpha \neq \beta$) modulo $\mathbb{Z} + \mathbb{Z}\tau$. Beacuse both $\text{LHS}(n, 1)$ and $\text{RHS}(n, 1)$ are symmetric with respect with u_1, u_2, \dots, u_{n+1} , it is enough to check the pole at $u_2 = u_1 - r$. We have

$$\begin{aligned} \text{Res}_{u_2=u_1-r} \text{LHS}(n, 1) &= \text{Res}_{u_2=u_1-r} \text{RHS}(n, 1) \\ &= \text{Res}_{u=0} \frac{[-r^*]_s [-1]_s}{[-r]_s [u]_s} \prod_{j=3}^{n+1} \frac{[u_j - u_1 + 1]_s [u_j - u_1 + r^*]_s}{[u_j - u_1]_s [u_j - u_1 + r]_s}. \end{aligned} \quad (3.75)$$

We have shown $n > m = 1$ case. We show general $n > m \geq 1$ case. We assume the equation $\text{LHS}(n, 1) = \text{RHS}(n, 1)$ for some (m, n) . Beacuse both $\text{LHS}(n+1, m+1)$ and $\text{RHS}(n+1, m+1)$ are symmetric with respect with $u_1, u_2, \dots, u_{n+m+2}$, it is enough to check the pole at $u_2 = u_1 - r$. Let us take the residue at $u_2 = u_1 - r$ for $(m+1, n+1)$.

$$\begin{aligned} \text{Res}_{u_2=u_1-r} & \left(\sum_{\substack{J \subset \{1, 2, \dots, n+m+2\} \\ |J|=n+1}} \prod_{j \in J} \prod_{k \notin J} \frac{[u_k - u_j + 1]_s [u_k - u_j + r^*]_s}{[u_k - u_j]_s [u_k - u_j + r]_s} \right. \\ & \left. - \sum_{\substack{J^c \subset \{1, 2, \dots, n+m+2\} \\ |J^c|=m+1}} \prod_{j \in J^c} \prod_{k \notin J^c} \frac{[u_k - u_j + 1]_s [u_k - u_j + r^*]_s}{[u_k - u_j]_s [u_k - u_j + r]_s} \right) \\ &= \prod_{j=3}^{n+m+2} \frac{[u_j - u_1 + 1]_s [u_j - u_1 + r^*]_s}{[u_j - u_1]_s [u_j - u_1 + r]_s} \\ & \times \left(\sum_{\substack{J \subset \{3, 4, \dots, n+m+2\} \\ |J|=n}} \prod_{j \in J} \prod_{k \notin J} \frac{[u_k - u_j + 1]_s [u_k - u_j + r^*]_s}{[u_k - u_j]_s [u_k - u_j + r]_s} \right. \\ & \left. - \sum_{\substack{J^c \subset \{3, 4, \dots, n+m+2\} \\ |J^c|=m}} \prod_{j \in J^c} \prod_{k \notin J^c} \frac{[u_k - u_j + 1]_s [u_k - u_j + r^*]_s}{[u_k - u_j]_s [u_k - u_j + r]_s} \right) = 0. \end{aligned} \quad (3.76)$$

We have used the assumption of induction $\text{LHS}(n, m) = \text{RHS}(n, m)$. Q.E.D.

Proposition 3.8 *The following weakly sense equation holds.*

$$\mathcal{O}_n(z_1, \dots, z_n) \mathcal{O}_m(z_{n+1}, \dots, z_{n+m}) \sim \prod_{\substack{1 \leq j \leq n \\ n+1 \leq k \leq n+m}} \frac{1}{g_{11}(z_k/z_j)} \mathcal{O}_{m+n}(z_1, \dots, z_{n+m}). \quad (3.77)$$

Proof This is direct consequence of the following explicit folrmulae

$$\begin{aligned} \mathcal{O}_{n+m}(z_1, \dots, z_{n+m}) &\sim \prod_{\substack{1 \leq j \leq n \\ n+1 \leq k \leq n+m}} g_{11}(z_k/z_j) \mathcal{O}_n(z_1, \dots, z_n) \mathcal{O}_m(z_{n+1}, \dots, z_{n+m}) \\ &+ \sum_{\substack{\alpha, \alpha_2, \dots, \alpha_N \geq 0 \\ \alpha_1 + 2\alpha_2 + \dots + N\alpha_N = n \\ 2 \leq 2\alpha_2 + \dots + N\alpha_N \leq n}} \prod_{t=1}^N \left((-c)^{t-1} \prod_{u=1}^{t-1} \Delta(x^{2u+1})^{t-u-1} \right)^{\alpha_t} \\ &\times \sum_{\substack{\{L_j^{(s)}\}_{s=1, \dots, N}, \{R_j^{(s)}\}_{s=1, \dots, N} \\ j=1, \dots, \alpha_s \\ \sum_{s=2}^N |L_1^{(s)}| |R_1^{(s)}| \geq 1}} \prod_{t=1}^N \prod_{j=1}^{\alpha_t} \sum_{\substack{\sigma \in S_t \\ \sigma(1)=1}} \prod_{u=1}^t \delta \left(\frac{x^2 z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}} \right) \\ &\times \prod_{t=1}^N \prod_{\substack{j < k \\ j, k \in A_{Min}^{(t)}}} g_{t,t} \left(\frac{z_k}{z_j} \right) \prod_{1 \leq t < u \leq N} \prod_{\substack{j \in A_{Min}^{(t)} \\ k \in A_{Min}^{(u)}}} g_{t,u} \left(x^{u-t-2[\frac{u}{2}]+2[\frac{t}{2}]} \frac{z_k}{z_j} \right). \end{aligned} \quad (3.78)$$

Here the summation $\sum_{\substack{\{L_j^{(s)}\}_{s=1, \dots, N}, \{R_j^{(s)}\}_{s=1, \dots, N} \\ j=1, \dots, \alpha_s \\ \sum_{s=2}^N |L_1^{(s)}| |R_1^{(s)}| \geq 1}}$ is taken over the conditions that

$$\begin{aligned} \cup_{s=1}^N \cup_{j=1}^{\alpha_s} L_j^{(s)} &= \{1, 2, \dots, m\}, L_i^{(s)} \cap L_j^{(s)} = \emptyset, (i \neq j), \\ \text{Min}(L_1^{(s)}) &< \text{Min}(L_2^{(s)}) < \dots < \text{Min}(L_{\alpha_s}^{(s)}), \\ \cup_{s=1}^N \cup_{j=1}^{\alpha_s} R_j^{(s)} &= \{m+1, m+2, \dots, m+n\}, R_i^{(s)} \cap R_j^{(s)} = \emptyset, (i \neq j), \\ \text{Min}(R_1^{(s)}) &< \text{Min}(R_2^{(s)}) < \dots < \text{Min}(R_{\alpha_s}^{(s)}). \end{aligned} \quad (3.79)$$

Here we have set $A_j^{(s)} = L_j^{(s)} \cup R_j^{(s)}$. We have set $A_{j,k}^{(s)} = j_k$ for $A_j^{(s)} = \{j_1 < j_2 < \dots < j_s\}$, and $A_{Min}^{(s)} = \{A_{1,1}^{(s)}, A_{2,1}^{(s)}, \dots, A_{\alpha_s,1}^{(s)}\}$. We want to point out that every term of the summation $\sum_{\substack{\{L_j^{(s)}\}_{s=1, \dots, N}, \{R_j^{(s)}\}_{s=1, \dots, N} \\ j=1, \dots, \alpha_s}} \delta(x^2 z_k/z_j)$, ($1 \leq j \leq m, m+1 \leq k \leq m+n$). Dividing $\prod_{\substack{1 \leq j \leq n \\ n+1 \leq k \leq n+m}} g_{11}(z_k/z_j)$ to both sides and using $1/g_{11}(x^{-2}) = 0$, we have this proposition . Q.E.D.

Proof of Theorem 3.1 At first we restrict ourself to the regime, $\text{Re}(s) > N$ and $\text{Re}(r) < 0$,

in order to use the power series formulae of the local integrals of motion, \mathcal{I}_n . In Proposition 3.5 we have shown for $\sigma \in S_n$

$$\prod_{1 \leq j < k \leq n} s(z_k/z_j) \mathcal{O}_n(z_1, \dots, z_n) = \prod_{1 \leq j < k \leq n} s(z_{\sigma(k)}/z_{\sigma(j)}) \mathcal{O}_n(z_{\sigma(1)}, \dots, z_{\sigma(n)}). \quad (3.80)$$

Hence we have

$$\begin{aligned} & \mathcal{I}_n \cdot \mathcal{I}_m \\ &= \left[\prod_{1 \leq j < k \leq n} s(z_k/z_j) \mathcal{O}_n(z_1, \dots, z_n) \prod_{n+1 \leq j < k \leq n+m} s(z_k/z_j) \mathcal{O}_m(z_{n+1}, \dots, z_{n+m}) \right]_{1, z_1 \dots z_{n+m}} \\ &= \left[\frac{1}{(n+m)!} \sum_{\sigma \in S_{n+m}} \prod_{j=1}^n \prod_{k=n+1}^{n+m} \frac{1}{h(u_{\sigma(k)} - u_{\sigma(j)})} \prod_{1 \leq j < k \leq n+m} s(z_k/z_j) \mathcal{O}_{n+m}(z_1, \dots, z_{n+m}) \right]_{1, z_1 \dots z_{n+m}}. \end{aligned} \quad (3.81)$$

Hence the commutation relation $\mathcal{I}_n \cdot \mathcal{I}_m = \mathcal{I}_m \cdot \mathcal{I}_n$ is reduced to the theta identity in Proposition 3.7.

$$\sum_{\sigma \in S_{m+n}} \prod_{j=1}^n \prod_{k=n+1}^{n+m} \frac{1}{h(u_{\sigma(k)} - u_{\sigma(j)})} = \sum_{\sigma \in S_{m+n}} \prod_{j=1}^m \prod_{k=m+1}^{n+m} \frac{1}{h(u_{\sigma(k)} - u_{\sigma(j)})}. \quad (3.82)$$

Proof of the commutation relation $[\mathcal{I}_m^*, \mathcal{I}_n^*] = 0$ is given as similar way. Here we omit details for \mathcal{I}_n^* . Q.E.D.

4 Nonlocal Integrals of Motion

In this section we give explicit formulae of the nonlocal integrals of motion. We study generic case : $0 < x < 1, \text{Re}(r) \neq 0$ and $s \in \mathbb{C}$ (resp. $0 < x < 1, \text{Re}(r^*) \neq 0$ and $s \in \mathbb{C}$).

4.1 Nonlocal Integrals of Motion

We explicitly construct the nonlocal integrals of motion and state the main results for $N \geq 3$. The results for $N = 2$ is summarized in [13].

Definition 4.1

- For the regime $\text{Re}(r) > 0$ and $0 < \text{Re}(s) < N$, we define a family of operators \mathcal{G}_m , ($m =$

$1, 2, \dots)$ by

$$\begin{aligned}
\mathcal{G}_m &= \prod_{t=1}^N \prod_{j=1}^m \oint_C \frac{dz_j^{(t)}}{2\pi\sqrt{-1}z_j^{(t)}} F_1(z_1^{(1)}) \cdots F_1(z_m^{(1)}) F_2(z_1^{(2)}) \cdots F_2(z_m^{(2)}) \cdots F_N(z_1^{(N)}) \cdots F_N(z_m^{(N)}) \\
&\quad \times \frac{\prod_{t=1}^N \prod_{1 \leq i < j \leq m} \left[u_i^{(t)} - u_j^{(t)} \right]_r \left[u_j^{(t)} - u_i^{(t)} - 1 \right]_r}{\prod_{t=1}^{N-1} \prod_{i,j=1}^m \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^m \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r} \\
&\quad \times \vartheta \left(\sum_{j=1}^m u_j^{(1)} \middle| \sum_{j=1}^m u_j^{(2)} \middle| \cdots \middle| \sum_{j=1}^m u_j^{(N)} \right). \tag{4.1}
\end{aligned}$$

Here we have set the theta function $\vartheta(u^{(1)}|u^{(2)}|\cdots|u^{(N)})$ by

$$\vartheta(u^{(1)}|\cdots|u^{(t)}+r|\cdots|u^{(N)}) = \vartheta(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}), \quad (1 \leq t \leq N) \tag{4.2}$$

$$\begin{aligned}
&\vartheta(u^{(1)}|\cdots|u^{(t)}+r\tau|\cdots|u^{(N)}) \\
&= e^{-2\pi i\tau + \frac{2\pi i}{r}(u_{t-1}-2u_t+u_{t+1}+\sqrt{r(r-1})P_{\alpha_t})} \vartheta(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}), \quad (1 \leq t \leq N), \tag{4.3}
\end{aligned}$$

$$\vartheta(u^{(1)}+k|\cdots|u^{(N)}+k) = \vartheta(u^{(1)}|\cdots|u^{(N)}), \quad (k \in \mathbb{C}), \tag{4.4}$$

$$\eta(\vartheta(u^{(1)}|\cdots|u^{(N)})) = \vartheta(u^{(N)}|u^{(1)}|\cdots|u^{(N-1)}). \tag{4.5}$$

Here the integral contour C is given by

$$|x^{\frac{2s}{N}} z_j^{(t+1)}| < |z_i^{(t)}| < |x^{-2+\frac{2s}{N}} z_j^{(t+1)}|, \quad (1 \leq t \leq N-1, 1 \leq i, j \leq m), \tag{4.6}$$

$$|x^{2-\frac{2s}{N}} z_j^{(1)}| < |z_i^{(N)}| < |x^{-\frac{2s}{N}} z_j^{(1)}|, \quad (1 \leq i, j \leq m). \tag{4.7}$$

For generic $s \in \mathbb{C}$, the definition of \mathcal{G}_n should be understood as analytic continuation.

We call the operator \mathcal{G}_n the nonlocal integrals of motion for the deformed W -algebra $W_{q,t}(\widehat{sl}_N)$.

- For the regime $\text{Re}(r) < 0$ and $0 < \text{Re}(s) < N$, we define a family of operators \mathcal{G}_m , ($m = 1, 2, \dots$) by

$$\begin{aligned}
\mathcal{G}_m &= \prod_{t=1}^N \prod_{j=1}^m \oint_C \frac{dz_j^{(t)}}{2\pi\sqrt{-1}z_j^{(t)}} F_1(z_1^{(1)}) \cdots F_1(z_m^{(1)}) F_2(z_1^{(2)}) \cdots F_2(z_m^{(2)}) \cdots F_N(z_1^{(N)}) \cdots F_N(z_m^{(N)}) \\
&\quad \times \frac{\prod_{t=1}^N \prod_{1 \leq i < j \leq m} \left[u_i^{(t)} - u_j^{(t)} \right]_{-r} \left[u_j^{(t)} - u_i^{(t)} + 1 \right]_{-r}}{\prod_{t=1}^{N-1} \prod_{i,j=1}^m \left[u_i^{(t)} - u_j^{(t+1)} - \frac{s}{N} \right]_{-r} \prod_{i,j=1}^m \left[u_i^{(1)} - u_j^{(N)} - 1 + \frac{s}{N} \right]_{-r}}
\end{aligned}$$

$$\times \vartheta \left(\sum_{j=1}^m u_j^{(1)} \middle| \sum_{j=1}^m u_j^{(2)} \middle| \cdots \middle| \sum_{j=1}^m u_j^{(N)} \right). \quad (4.8)$$

Here we have set the theta function $\vartheta(u^{(1)}|u^{(2)}|\cdots|u^{(N)})$ by

$$\vartheta(u^{(1)}|\cdots|u^{(t)}+r|\cdots|u^{(N)}) = \vartheta(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}), \quad (1 \leq t \leq N) \quad (4.9)$$

$$\begin{aligned} & \vartheta(u^{(1)}|\cdots|u^{(t)}-r\tau|\cdots|u^{(N)}) \\ = & e^{-2\pi i\tau - \frac{2\pi i}{\tau}(u_{t-1}-2u_t+u_{t+1}+\sqrt{r(r-1)}P_{\alpha_t})} \vartheta(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}), \quad (1 \leq t \leq N) \end{aligned} \quad (4.10)$$

$$\vartheta(u^{(1)}+k|\cdots|u^{(N)}+k) = \vartheta(u^{(1)}|\cdots|u^{(N)}), \quad (k \in \mathbb{C}), \quad (4.11)$$

$$\eta(\vartheta(u^{(1)}|\cdots|u^{(N)})) = \vartheta(u^{(N)}|u^{(1)}|\cdots|u^{(N-1)}). \quad (4.12)$$

Here the integral contour C is given by

$$|x^{-2+\frac{2s}{N}} z_j^{(t+1)}| < |z_i^{(t)}| < |x^{\frac{2s}{N}} z_j^{(t+1)}|, \quad (1 \leq t \leq N-1, 1 \leq i, j \leq m), \quad (4.13)$$

$$|x^{-\frac{2s}{N}} z_j^{(1)}| < |z_i^{(N)}| < |x^{2-\frac{2s}{N}} z_j^{(1)}|, \quad (1 \leq i, j \leq m). \quad (4.14)$$

For generic $s \in \mathbb{C}$, the definition of \mathcal{G}_n should be understood as analytic continuation. We call the operator \mathcal{G}_n the nonlocal integrals of motion for the deformed W -algebra $W_{q,t}(\widehat{sl}_N)$.

- For $\text{Re}(r^*) > 0$ and $0 < \text{Re}(s) < N$, we define a family of operators \mathcal{G}_m^* , ($m = 1, 2, \dots$) by

$$\begin{aligned} \mathcal{G}_m^* &= \prod_{t=1}^N \prod_{j=1}^m \oint_{C^*} \frac{dz_j^{(t)}}{2\pi\sqrt{-1}z_j^{(t)}} E_1(z_1^{(1)}) \cdots E_1(z_m^{(1)}) E_2(z_1^{(2)}) \cdots E_2(z_m^{(2)}) \cdots E_N(z_1^{(N)}) \cdots E_N(z_m^{(N)}) \\ &\quad \times \frac{\prod_{t=1}^N \prod_{1 \leq i < j \leq m} [u_i^{(t)} - u_j^{(t)}]_{r^*} [u_j^{(t)} - u_i^{(t)} + 1]_{r^*}}{\prod_{t=1}^{N-1} \prod_{i,j=1}^m \left[u_i^{(t)} - u_j^{(t+1)} - \frac{s}{N} \right]_{r^*} \prod_{i,j=1}^m \left[u_i^{(1)} - u_j^{(N)} - 1 + \frac{s}{N} \right]_{r^*}} \\ &\quad \times \vartheta^* \left(\sum_{j=1}^m u_j^{(1)} \middle| \sum_{j=1}^m u_j^{(2)} \middle| \cdots \middle| \sum_{j=1}^m u_j^{(N)} \right). \end{aligned} \quad (4.15)$$

Here we have set the theta function $\vartheta^*(u^{(1)}|u^{(2)}|\cdots|u^{(N)})$ by

$$\vartheta^*(u^{(1)}|\cdots|u^{(t)}+r|\cdots|u^{(N)}) = \vartheta^*(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}), \quad (1 \leq t \leq N) \quad (4.16)$$

$$\begin{aligned} & \vartheta^*(u^{(1)}|\cdots|u^{(t)}+r^*\tau|\cdots|u^{(N)}) \\ = & e^{-2\pi i\tau + \frac{2\pi i}{r^*}(u_{t-1}-2u_t+u_{t+1}+\sqrt{r(r-1)}P_{\alpha_t})} \vartheta^*(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}), \quad (1 \leq t \leq N) \end{aligned} \quad (4.17)$$

$$\vartheta^*(u^{(1)}+k|\cdots|u^{(N)}+k) = \vartheta^*(u^{(1)}|\cdots|u^{(N)}), \quad (k \in \mathbb{C}), \quad (4.18)$$

$$\eta(\vartheta^*(u^{(1)}|\cdots|u^{(N)})) = \vartheta^*(u^{(N)}|u^{(1)}|\cdots|u^{(N-1)}). \quad (4.19)$$

Here the integral contour C^* is given by

$$|x^{-2+\frac{2s}{N}} z_j^{(t+1)}| < |z_i^{(t)}| < |x^{\frac{2s}{N}} z_j^{(t+1)}|, \quad (1 \leq t \leq N-1, 1 \leq i, j \leq m), \quad (4.20)$$

$$|x^{-\frac{2s}{N}} z_j^{(1)}| < |z_i^{(N)}| < |x^{2-\frac{2s}{N}} z_j^{(1)}|, \quad (1 \leq i, j \leq m). \quad (4.21)$$

For generic $s \in \mathbb{C}$, the definition of \mathcal{G}_n should be understood as analytic continuation. We call the operator \mathcal{G}_n the nonlocal integrals of motion for the deformed W -algebra $W_{q,t}(\widehat{sl}_N)$.

- For $\text{Re}(r^*) < 0$ and $0 < \text{Re}(s) < N$, we define a family of operators \mathcal{G}_m^* , ($m = 1, 2, \dots$) by

$$\begin{aligned} \mathcal{G}_m^* &= \prod_{t=1}^N \prod_{j=1}^m \oint_{C^*} \frac{dz_j^{(t)}}{2\pi\sqrt{-1}z_j^{(t)}} E_1(z_1^{(1)}) \cdots E_1(z_m^{(1)}) E_2(z_1^{(2)}) \cdots E_2(z_m^{(2)}) \cdots E_N(z_1^{(N)}) \cdots E_N(z_m^{(N)}) \\ &\quad \times \frac{\prod_{t=1}^N \prod_{1 \leq i < j \leq m} [u_i^{(t)} - u_j^{(t)}]_{-r^*} [u_j^{(t)} - u_i^{(t)} - 1]_{-r^*}}{\prod_{t=1}^{N-1} \prod_{i,j=1}^m \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_{-r^*} \prod_{i,j=1}^m \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_{-r^*}} \\ &\quad \times \vartheta^* \left(\sum_{j=1}^m u_j^{(1)} \middle| \sum_{j=1}^m u_j^{(2)} \middle| \cdots \middle| \sum_{j=1}^m u_j^{(N)} \right). \end{aligned} \quad (4.22)$$

Here we have set the theta function $\vartheta^*(u^{(1)}|u^{(2)}|\cdots|u^{(N)})$ by

$$\vartheta^*(u^{(1)}|\cdots|u^{(t)}+r|\cdots|u^{(N)}) = \vartheta^*(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}), \quad (1 \leq t \leq N) \quad (4.23)$$

$$\begin{aligned} &\vartheta^*(u^{(1)}|\cdots|u^{(t)}r^*\tau|\cdots|u^{(N)}) \\ &= e^{-2\pi i\tau - \frac{2\pi i}{r^*}(u_{t-1}-2u_t+u_{t+1}+\sqrt{r(r-1)}P_{\alpha_t})} \vartheta^*(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}), \quad (1 \leq t \leq N) \end{aligned} \quad (4.24)$$

$$\vartheta^*(u^{(1)}+k|\cdots|u^{(N)}+k) = \vartheta^*(u^{(1)}|\cdots|u^{(N)}), \quad (k \in \mathbb{C}), \quad (4.25)$$

$$\eta(\vartheta^*(u^{(1)}|\cdots|u^{(N)})) = \vartheta^*(u^{(N)}|u^{(1)}|\cdots|u^{(N-1)}). \quad (4.26)$$

Here the integral contour C^* is given by

$$|x^{\frac{2s}{N}} z_j^{(t+1)}| < |z_i^{(t)}| < |x^{-2+\frac{2s}{N}} z_j^{(t+1)}|, \quad (1 \leq t \leq N-1, 1 \leq i, j \leq m), \quad (4.27)$$

$$|x^{2-\frac{2s}{N}} z_j^{(1)}| < |z_i^{(N)}| < |x^{-\frac{2s}{N}} z_j^{(1)}|, \quad (1 \leq i, j \leq m). \quad (4.28)$$

For generic $s \in \mathbb{C}$, the definition of \mathcal{G}_n should be understood as analytic continuation. We call the operator \mathcal{G}_n the nonlocal integrals of motion for the deformed W -algebra $W_{q,t}(\widehat{sl}_N)$.

We summarize explicit formulae for the integrand function $\vartheta(u^{(1)}|u^{(2)}|\cdots|u^{(N)})$.

Proposition 4.1 For $\alpha_1, \alpha_2, \dots, \alpha_N \in \mathbb{C}$ and $\text{Re}(r) > 0$, we set the theta function $\tilde{\vartheta}_\alpha(u^{(1)}|u^{(2)}|\dots|u^{(N)})$ by

$$\begin{aligned} \tilde{\vartheta}_\alpha(u^{(1)}|u^{(2)}|\dots|u^{(N)}) &= [u^{(1)} - u^{(2)} - \sqrt{rr^*}P_{\bar{\epsilon}_2} + \alpha_1 P_{\bar{\epsilon}_1} + \alpha_2 P_{\bar{\epsilon}_2} + \dots + \alpha_N P_{\bar{\epsilon}_N}]_r \\ &\times [u^{(2)} - u^{(3)} - \sqrt{rr^*}P_{\bar{\epsilon}_3} + \alpha_1 P_{\bar{\epsilon}_1} + \alpha_2 P_{\bar{\epsilon}_2} + \dots + \alpha_N P_{\bar{\epsilon}_N}]_r \\ &\times \dots \\ &\times [u^{(N)} - u^{(1)} - \sqrt{rr^*}P_{\bar{\epsilon}_1} + \alpha_1 P_{\bar{\epsilon}_1} + \alpha_2 P_{\bar{\epsilon}_2} + \dots + \alpha_N P_{\bar{\epsilon}_N}]_r. \end{aligned} \quad (4.29)$$

This theta function $\tilde{\vartheta}_\alpha(u^{(1)}|\dots|u^{(N)})$ satisfies the conditions

$$\tilde{\vartheta}_\alpha(u^{(1)}|\dots|u^{(t)}+r|\dots|u^{(N)}) = \tilde{\vartheta}_\alpha(u^{(1)}|\dots|u^{(t)}|\dots|u^{(N)}), \quad (1 \leq t \leq N) \quad (4.30)$$

$$\begin{aligned} &\tilde{\vartheta}_\alpha(u^{(1)}|\dots|u^{(t)}+r\tau|\dots|u^{(N)}) \\ &= e^{-2\pi i\tau + \frac{2\pi i}{r}(u_{t-1}-2u_t+u_{t+1}+\sqrt{r(r-1)}P_{\alpha_t})} \tilde{\vartheta}_\alpha(u^{(1)}|\dots|u^{(t)}|\dots|u^{(N)}), \quad (1 \leq t \leq N) \end{aligned} \quad (4.31)$$

$$\tilde{\vartheta}_\alpha(u^{(1)}+k|\dots|u^{(N)}+k) = \tilde{\vartheta}_\alpha(u^{(1)}|\dots|u^{(N)}), \quad (k \in \mathbb{C}), \quad (4.32)$$

$$\eta(\tilde{\vartheta}_\alpha(u^{(1)}|\dots|u^{(N)})) = \tilde{\vartheta}_\alpha(u^{(N)}|u^{(1)}|\dots|u^{(N-1)}). \quad (4.33)$$

Proof Let us set $\tilde{\vartheta}(u^{(1)}|\dots|u^{(N)}) = [u^{(1)} - u^{(2)} + \hat{\pi}_{1,2}]_r [u^{(2)} - u^{(3)} + \hat{\pi}_{2,3}]_r \dots [u^{(N)} - u^{(1)} + \hat{\pi}_{N,1}]_r$. The second quasi-periodic condition is equivalent with

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & -1 \\ -1 & 1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{\pi}_{1,2} \\ \hat{\pi}_{2,3} \\ \hat{\pi}_{1,2} \\ \dots \\ \hat{\pi}_{N,1} \end{pmatrix} = \sqrt{rr^*} \begin{pmatrix} P_{\alpha_1} \\ P_{\alpha_2} \\ P_{\alpha_3} \\ \dots \\ P_{\alpha_N} \end{pmatrix}. \quad (4.34)$$

Hence we have the general solution for $\hat{\pi}_{i,i+1}$ by

$$\begin{pmatrix} \hat{\pi}_{1,2} \\ \hat{\pi}_{2,3} \\ \dots \\ \hat{\pi}_{N,1} \end{pmatrix} = \begin{pmatrix} -\sqrt{rr^*}P_{\bar{\epsilon}_2} + \alpha_1 P_{\bar{\epsilon}_1} + \alpha_2 P_{\bar{\epsilon}_2} + \dots + \alpha_N P_{\bar{\epsilon}_N} \\ -\sqrt{rr^*}P_{\bar{\epsilon}_3} + \alpha_1 P_{\bar{\epsilon}_1} + \alpha_2 P_{\bar{\epsilon}_2} + \dots + \alpha_N P_{\bar{\epsilon}_N} \\ \dots \\ -\sqrt{rr^*}P_{\bar{\epsilon}_1} + \alpha_1 P_{\bar{\epsilon}_1} + \alpha_2 P_{\bar{\epsilon}_2} + \dots + \alpha_N P_{\bar{\epsilon}_N} \end{pmatrix}, \quad (4.35)$$

where $\alpha_1, \alpha_2, \dots, \alpha_N \in \mathbb{C}$. Other conditions are trivial. Q.E.D.

Example For $N = 2$, $m = 1$ and $\text{Re}(r) > 0$ case, we have

$$\mathcal{G}_1 = \int \int_C \frac{dz_1}{2\pi\sqrt{-1}z_1} \frac{dz_2}{2\pi\sqrt{-1}z_2} F_1(z_1) F_2(z_2) \frac{\vartheta(u_1|u_2)}{[u_1 - u_2 + \frac{s}{2}]_r [u_1 - u_2 - \frac{s}{2} + 1]_r}. \quad (4.36)$$

Here C is given by

$$|x^s z_2| < |z_1| < |x^{-2+s} z_2|.$$

Example For $N = 3$, $m = 1$ and $\text{Re}(r) > 0$ case, we have

$$\begin{aligned} \mathcal{G}_1 &= \int \int \int_C \frac{dz_1}{2\pi\sqrt{-1}z_1} \frac{dz_2}{2\pi\sqrt{-1}z_2} \frac{dz_3}{2\pi\sqrt{-1}z_3} F_1(z_1) F_2(z_2) F_3(z_3) \\ &\times \frac{\vartheta(u_1|u_2|u_3)}{[u_1 - u_2 + 1 - \frac{s}{3}]_r [u_2 - u_3 + 1 - \frac{s}{3}]_r [u_1 - u_3 + \frac{s}{3}]_r}. \end{aligned} \quad (4.37)$$

Here C is given by

$$|x^{\frac{2s}{3}} z_2| < |z_1| < |x^{-2+\frac{2s}{3}} z_2|, |x^{\frac{2s}{3}} z_3| < |z_2| < |x^{-2+\frac{2s}{3}} z_3|, |x^{2-\frac{2s}{3}} z_1| < |z_3| < |x^{-\frac{2s}{3}} z_1|.$$

The followings are some of **Main Results**.

Theorem 4.2 *The nonlocal integrals of motion \mathcal{G}_n commute with each other.*

$$[\mathcal{G}_m, \mathcal{G}_n] = 0, \quad (m, n = 1, 2, \dots). \quad (4.38)$$

The nonlocal integrals of motion \mathcal{G}_n^ commute with each other.*

$$[\mathcal{G}_m^*, \mathcal{G}_n^*] = 0, \quad (m, n = 1, 2, \dots). \quad (4.39)$$

Theorem 4.3 *The nonlocal integrals of motion \mathcal{G}_n and \mathcal{G}_n^* commute with each other for regime $0 < \text{Re}(r)$ and $\text{Re}(r^*) < 0$.*

$$[\mathcal{G}_m, \mathcal{G}_n^*] = 0, \quad (m, n = 1, 2, \dots). \quad (4.40)$$

Theorem 4.4 *The local integrals of motion \mathcal{I}_n , \mathcal{I}_n^* and nonlocal integrals of motion \mathcal{G}_m , \mathcal{G}_m^* commute with each other.*

$$[\mathcal{I}_n, \mathcal{G}_m] = 0, \quad [\mathcal{I}_n, \mathcal{G}_m^*] = 0, \quad (m, n = 1, 2, \dots), \quad (4.41)$$

$$[\mathcal{I}_n^*, \mathcal{G}_m] = 0, \quad [\mathcal{I}_n^*, \mathcal{G}_m^*] = 0, \quad (m, n = 1, 2, \dots). \quad (4.42)$$

4.2 Proof of $[\mathcal{G}_m, \mathcal{G}_n] = 0$

In this section we study the commutation relations $[\mathcal{G}_m, \mathcal{G}_n] = 0$ for $\text{Re}(r) > 0$. We omit details for other cases, because they are similar.

Proposition 4.5 *For $\text{Re}(r) > 0$ we have*

$$\begin{aligned}
& \sum_{\sigma_1 \in S_{m+n}} \sum_{\sigma_2 \in S_{m+n}} \cdots \sum_{\sigma_N \in S_{m+n}} \hat{\vartheta}_\alpha \left(\sum_{j=1}^m u_{\sigma_1(j)}^{(1)} \left| \sum_{j=1}^m u_{\sigma_2(j)}^{(2)} \right| \cdots \left| \sum_{j=1}^m u_{\sigma_N(j)}^{(N)} \right| \right) \\
& \times \hat{\vartheta}_\beta \left(\sum_{j=m+1}^{m+n} u_{\sigma_1(j)}^{(1)} \left| \sum_{j=m+1}^{m+n} u_{\sigma_2(j)}^{(2)} \right| \cdots \left| \sum_{j=m+1}^{m+n} u_{\sigma_N(j)}^{(N)} \right| \right) \\
& \times \prod_{t=1}^N \prod_{i=1}^m \prod_{j=m+1}^{m+n} \frac{\left[u_{\sigma_t(i)}^{(t)} - u_{\sigma_{t+1}(j)}^{(t+1)} - \frac{s}{N} \right]_r \left[u_{\sigma_{t+1}(i)}^{(t+1)} - u_{\sigma_t(j)}^{(t)} + 1 - \frac{s}{N} \right]_r}{\left[u_{\sigma_t(i)}^{(t)} - u_{\sigma_t(j)}^{(t)} \right]_r \left[u_{\sigma_t(j)}^{(t)} - u_{\sigma_t(i)}^{(t)} - 1 \right]_r} \\
& = \sum_{\sigma_1 \in S_{m+n}} \sum_{\sigma_2 \in S_{m+n}} \cdots \sum_{\sigma_N \in S_{m+n}} \hat{\vartheta}_\beta \left(\sum_{j=1}^n u_{\sigma_1(j)}^{(1)} \left| \sum_{j=1}^n u_{\sigma_2(j)}^{(2)} \right| \cdots \left| \sum_{j=1}^n u_{\sigma_N(j)}^{(N)} \right| \right) \\
& \times \hat{\vartheta}_\alpha \left(\sum_{j=n+1}^{m+n} u_{\sigma_1(j)}^{(1)} \left| \sum_{j=n+1}^{m+n} u_{\sigma_2(j)}^{(2)} \right| \cdots \left| \sum_{j=n+1}^{m+n} u_{\sigma_N(j)}^{(N)} \right| \right) \\
& \times \prod_{t=1}^N \prod_{i=1}^n \prod_{j=n+1}^{m+n} \frac{\left[u_{\sigma_t(i)}^{(t)} - u_{\sigma_{t+1}(j)}^{(t+1)} - \frac{s}{N} \right]_r \left[u_{\sigma_{t+1}(i)}^{(t+1)} - u_{\sigma_t(j)}^{(t)} + 1 - \frac{s}{N} \right]_r}{\left[u_{\sigma_t(i)}^{(t)} - u_{\sigma_t(j)}^{(t)} \right]_r \left[u_{\sigma_t(j)}^{(t)} - u_{\sigma_t(i)}^{(t)} - 1 \right]_r}. \tag{4.43}
\end{aligned}$$

Here $\hat{\vartheta}_\alpha(u^{(1)}|u^{(2)}|\cdots|u^{(N)})$ and $\hat{\vartheta}_\beta(u^{(1)}|u^{(2)}|\cdots|u^{(N)})$ are given by

$$\hat{\vartheta}_\alpha(u^{(1)}|\cdots|u^{(t)}+r|\cdots|u^{(N)}) = \hat{\vartheta}_\alpha(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}), \quad (1 \leq t \leq N) \tag{4.44}$$

$$\hat{\vartheta}_\alpha(u^{(1)}|\cdots|u^{(t)}+r\tau|\cdots|u^{(N)}) \tag{4.45}$$

$$= e^{-2\pi i\tau + \frac{2\pi i}{r}(u_{t-1}-2u_t+u_{t+1}+\sqrt{r(r-1)}P_{\alpha_t})+\nu_{\alpha,t}} \hat{\vartheta}_\alpha(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}), \quad (1 \leq t \leq N),$$

$$\hat{\vartheta}_\beta(u^{(1)}|\cdots|u^{(t)}+r|\cdots|u^{(N)}) = \hat{\vartheta}_\beta(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}), \quad (1 \leq t \leq N) \tag{4.46}$$

$$\hat{\vartheta}_\beta(u^{(1)}|\cdots|u^{(t)}+r\tau|\cdots|u^{(N)}) \tag{4.47}$$

$$= e^{-2\pi i\tau + \frac{2\pi i}{r}(u_{t-1}-2u_t+u_{t+1}+\sqrt{r(r-1)}P_{\alpha_t})+\nu_{\beta,t}} \hat{\vartheta}_\beta(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}), \quad (1 \leq t \leq N).$$

Here $\nu_{\alpha,t}, \nu_{\beta,t} \in \mathbb{C}$, $(1 \leq t \leq N)$.

Proof. In order to consider elliptic function, we divide the above theta identity by $\hat{\vartheta}_\gamma(\sum_{j=1}^{m+n} u_j^{(1)}|\cdots|\sum_{j=1}^{n+m} u_j^{(N)})$ with $\nu_{\gamma,t} \in \mathbb{C}$, $(1 \leq t \leq N)$:

$$\hat{\vartheta}_\gamma(u^{(1)}|\cdots|u^{(t)}+r|\cdots|u^{(N)}) = \hat{\vartheta}_\gamma(u^{(1)}|\cdots|u^{(t)}|\cdots|u^{(N)}), \quad (1 \leq t \leq N) \tag{4.48}$$

$$\begin{aligned}
& \widehat{\vartheta}_\gamma(u^{(1)} | \dots | u^{(t)} + r\tau | \dots | u^{(N)}) \\
&= e^{-2\pi i \tau + \frac{2\pi i}{r}(u_{t-1} - 2u_t + u_{t+1} + \sqrt{r(r-1)}P_{\alpha_t}) + \nu_{\gamma,t}} \widehat{\vartheta}_\gamma(u^{(1)} | \dots | u^{(t)} | \dots | u^{(N)}), \quad (1 \leq t \leq N).
\end{aligned} \tag{4.49}$$

Let us set

$$\begin{aligned}
\text{LHS}(m, n) &= \sum_{\substack{K_1 \cup K_1^c = \{1, 2, \dots, n+m\} \\ |K_1|=m, |K_1^c|=n}} \dots \sum_{\substack{K_N \cup K_N^c = \{1, 2, \dots, n+m\} \\ |K_N|=m, |K_N^c|=n}} \\
&\times \frac{\widehat{\vartheta}_\alpha \left(\sum_{j \in K_1} u_j^{(1)} \middle| \dots \middle| \sum_{j \in K_N} u_j^{(N)} \right) \widehat{\vartheta}_\beta \left(\sum_{j \in K_1^c} u_j^{(1)} \middle| \dots \middle| \sum_{j \in K_N^c} u_j^{(N)} \right)}{\widehat{\vartheta}_\gamma \left(\sum_{j=1}^{m+n} u_j^{(1)} \middle| \dots \middle| \sum_{j=1}^{m+n} u_j^{(N)} \right)} \\
&\times \prod_{t=1}^N \frac{\prod_{i \in K_t} \prod_{j \in K_{t+1}^c} \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i \in K_{t+1}} \prod_{j \in K_t^c} \left[u_i^{(t+1)} - u_j^{(t)} + \frac{s}{N} \right]_r}{\prod_{i \in K_t} \prod_{j \in K_t^c} \left[u_i^{(t)} - u_j^{(t)} \right]_r \left[u_j^{(t)} - u_i^{(t)} - 1 \right]_r},
\end{aligned} \tag{4.50}$$

$$\begin{aligned}
\text{RHS}(m, n) &= \sum_{\substack{K_1 \cup K_1^c = \{1, 2, \dots, n+m\} \\ |K_1|=n, |K_1^c|=m}} \dots \sum_{\substack{K_N \cup K_N^c = \{1, 2, \dots, n+m\} \\ |K_N|=n, |K_N^c|=m}} \\
&\times \frac{\widehat{\vartheta}_\beta \left(\sum_{j \in K_1} u_j^{(1)} \middle| \dots \middle| \sum_{j \in K_N} u_j^{(N)} \right) \widehat{\vartheta}_\alpha \left(\sum_{j \in K_1^c} u_j^{(1)} \middle| \dots \middle| \sum_{j \in K_N^c} u_j^{(N)} \right)}{\widehat{\vartheta}_\gamma \left(\sum_{j=1}^{m+n} u_j^{(1)} \middle| \dots \middle| \sum_{j=1}^{m+n} u_j^{(N)} \right)} \\
&\times \prod_{t=1}^N \frac{\prod_{i \in K_t} \prod_{j \in K_{t+1}^c} \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i \in K_{t+1}} \prod_{j \in K_t^c} \left[u_i^{(t+1)} - u_j^{(t)} + \frac{s}{N} \right]_r}{\prod_{i \in K_t} \prod_{j \in K_t^c} \left[u_i^{(t)} - u_j^{(t)} \right]_r \left[u_j^{(t)} - u_i^{(t)} - 1 \right]_r},
\end{aligned} \tag{4.51}$$

Candidates of poles of both $\text{LHS}(m, n)$ and $\text{RHS}(m, n)$ are $u_i^{(t)} = u_j^{(t)}$ and $u_i^{(t)} = u_j^{(t)} + 1$ and $\vartheta_\gamma = 0$. Let us show that the points $u_i^{(t)} = u_j^{(t)}$ are regular. Take the residue of the $\text{LHS}(m, n)$ at $u_1^{(1)} = u_2^{(1)}$. We have

$$\begin{aligned}
& \text{Res}_{u_1^{(1)}=u_2^{(1)}} \left(\frac{1}{[u_1^{(1)} - u_2^{(1)}]_r [u_2^{(1)} - u_1^{(1)} - 1]_r} + \frac{1}{[u_1^{(1)} - u_2^{(1)}]_r [u_1^{(1)} - u_2^{(1)} - 1]_r} \right) \\
&\times \sum_{\substack{L_1 \cup L_1^c = \{3, 4, \dots, n+m\} \\ |L_1|=m-1, |L_1^c|=n-1}} \sum_{\substack{K_2 \cup K_2^c = \{1, 2, \dots, n+m\} \\ |K_1|=m, |K_1^c|=n}} \dots \sum_{\substack{K_N \cup K_N^c = \{1, 2, \dots, n+m\} \\ |K_N|=m, |K_N^c|=n}}
\end{aligned}$$

$$\begin{aligned}
& \widehat{\vartheta}_\alpha \left(\sum_{j \in L_1 \cup \{1\}} u_j^{(1)} \mid \cdots \mid \sum_{j \in K_N} u_j^{(N)} \right) \widehat{\vartheta}_\beta \left(\sum_{j \in L_1^c \cup \{1\}} u_j^{(1)} \mid \cdots \mid \sum_{j \in K_N^c} u_j^{(N)} \right) \\
& \times \frac{\widehat{\vartheta}_\gamma \left(\sum_{j=1}^{m+n} u_j^{(1)} \mid \cdots \mid \sum_{j=1}^{n+m} u_j^{(N)} \right)}{\prod_{j \in K_2^c} \left[u_1^{(1)} - u_j^{(2)} + 1 - \frac{s}{N} \right]_r \prod_{i \in K_2} \left[u_i^{(2)} - u_2^{(1)} + \frac{s}{N} \right]_r} \\
& \times \frac{\prod_{j \in L_1^c} \left[u_1^{(1)} - u_j^{(1)} \right]_r \left[u_j^{(1)} - u_1^{(1)} - 1 \right]_r}{\prod_{j \in K_N} \left[u_i^{(N)} - u_2^{(1)} + 1 - \frac{s}{N} \right]_r \prod_{j \in K_N^c} \left[u_1^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r} \\
& \times \frac{\prod_{j \in L_1} \left[u_i^{(1)} - u_2^{(1)} \right]_r \left[u_2^{(1)} - u_i^{(1)} - 1 \right]_r}{\prod_{i \in L_1} \prod_{j \in K_2^c} \left[u_i^{(1)} - u_j^{(2)} + 1 - \frac{s}{N} \right]_r \prod_{i \in K_2} \prod_{j \in L_1^c} \left[u_i^{(2)} - u_j^{(1)} + \frac{s}{N} \right]_r} \\
& \times \frac{\prod_{i \in L_1} \prod_{j \in L_1^c} \left[u_i^{(1)} - u_j^{(1)} \right]_r}{\prod_{j \in K_N} \prod_{j \in L_1^c} \left[u_i^{(N)} - u_j^{(1)} + 1 - \frac{s}{N} \right]_r \prod_{i \in L_1} \prod_{j \in K_N^c} \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r} \\
& \times \frac{\prod_{i \in L_1} \prod_{j \in L_1^c} \left[u_j^{(1)} - u_i^{(1)} - 1 \right]_r}{\prod_{i \in K_t} \prod_{j \in K_{t+1}^c} \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i \in K_{t+1}} \prod_{j \in K_t^c} \left[u_i^{(t+1)} - u_j^{(t)} + \frac{s}{N} \right]_r} \\
& \times \frac{\prod_{t=2}^N \prod_{i \in K_t} \prod_{j \in K_{t+1}^c} \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i \in K_{t+1}} \prod_{j \in K_t^c} \left[u_i^{(t+1)} - u_j^{(t)} + \frac{s}{N} \right]_r}{\prod_{i \in K_t} \prod_{j \in K_t^c} \left[u_i^{(t)} - u_j^{(t)} \right]_r \left[u_j^{(t)} - u_i^{(t)} - 1 \right]_r} = 0.
\end{aligned} \tag{4.52}$$

Because the first term : $\text{Res}_{u_1^{(1)}=u_2^{(1)}} \left(\frac{1}{[u_1^{(1)}-u_2^{(1)}]_r [u_2^{(1)}-u_1^{(1)}-1]_r} + \frac{1}{[u_1^{(1)}-u_2^{(1)}]_r [u_1^{(1)}-u_2^{(1)}-1]_r} \right) = 0$, we have $\text{Res}_{u_1^{(1)}=u_2^{(1)}} \text{LHS}(m, n) = 0$. Because $\text{LHS}(m, n)$ is symmetric with respect to variables $u_1^{(1)}, u_2^{(1)}, \dots, u_{m+n}^{(1)}$, we have $\text{Res}_{u_i^{(1)}=u_j^{(1)}} \text{LHS}(m, n) = 0$ for $1 \leq i \neq j \leq m+n$. As the same manner as above, we conclude that points $u_i^{(t)} = u_j^{(t)}$ of $\text{LHS}(m, n)$ and $\text{RHS}(m, n)$ for $1 \leq t \leq N$, $1 \leq i \neq j \leq m+n$ are regular. Let us show $\text{LHS}(m, n) = \text{RHS}(m, n)$ by induction for $m+n$. Candidates of poles are only $u_i^{(t)} = u_j^{(t)} + 1$, $1 \leq t \leq N$ and $1 \leq i \neq j \leq m+n$. We assume $1 \leq m < n$ without losing generality. (The case $m = n$ is trivial.) At first we show the starting point $1 = m < n$: $\text{LHS}(1, n) = \text{RHS}(1, n)$. By straightfoeward calculations, we have

$$\text{Res}_{u_2^{(1)}=u_1^{(1)}+1} \cdots \text{Res}_{u_2^{(N)}=u_1^{(N)}+1} \text{LHS}(1, n)$$

$$\begin{aligned}
&= \prod_{t=1}^N \text{Res}_{u_2^{(t)}=u_1^{(t)}+1} \frac{\left[u_1^{(t)} - u_2^{(t+1)} + 1 - \frac{s}{N} \right]_r \left[u_1^{(t+1)} - u_2^{(t)} + \frac{s}{N} \right]_r}{\left[u_1^{(t)} - u_2^{(t)} \right]_r \left[u_2^{(t)} - u_1^{(t)} - 1 \right]_r} \\
&\times \prod_{t=1}^N \prod_{j=3}^{n+1} \frac{\left[u_1^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \left[u_1^{(t+1)} - u_j^{(t)} + \frac{s}{N} \right]_r}{\left[u_1^{(t)} - u_j^{(t)} \right]_r \left[u_j^{(t)} - u_1^{(t)} - 1 \right]_r} \\
&\times \frac{\widehat{\vartheta}_\alpha(u_1^{(1)} | \cdots | u_1^{(N)}) \widehat{\vartheta}_\beta \left(\sum_{j=2}^{n+1} u_j^{(1)} \middle| \cdots \middle| \sum_{j=2}^{n+1} u_j^{(N)} \right)}{\widehat{\vartheta}_\gamma \left(\sum_{j=1}^{n+1} u_j^{(1)} \middle| \cdots \middle| \sum_{j=1}^{n+1} u_j^{(N)} \right)}, \tag{4.53}
\end{aligned}$$

$$\begin{aligned}
&\text{Res}_{u_2^{(1)}=u_1^{(1)}+1} \cdots \text{Res}_{u_2^{(N)}=u_1^{(N)}+1} \text{RHS}(1, n) \\
&= \prod_{t=1}^N \text{Res}_{u_2^{(t)}=u_1^{(t)}+1} \frac{\left[u_1^{(t)} - u_2^{(t+1)} + 1 - \frac{s}{N} \right]_r \left[u_1^{(t+1)} - u_2^{(t)} + \frac{s}{N} \right]_r}{\left[u_1^{(t)} - u_2^{(t)} \right]_r \left[u_2^{(t)} - u_1^{(t)} - 1 \right]_r} \\
&\times \prod_{t=1}^N \prod_{i=3}^{n+1} \frac{\left[u_i^{(t)} - u_2^{(t+1)} + 1 - \frac{s}{N} \right]_r \left[u_i^{(t+1)} - u_2^{(t)} + \frac{s}{N} \right]_r}{\left[u_i^{(t)} - u_2^{(t)} \right]_r \left[u_2^{(t)} - u_i^{(t)} - 1 \right]_r} \\
&\times \frac{\widehat{\vartheta}_\alpha(u_2^{(1)} | \cdots | u_2^{(N)}) \widehat{\vartheta}_\beta \left(\sum_{\substack{j=1 \\ j \neq 2}}^{n+1} u_j^{(1)} \middle| \cdots \middle| \sum_{\substack{j=1 \\ j \neq 2}}^{n+1} u_j^{(N)} \right)}{\widehat{\vartheta}_\gamma \left(\sum_{j=1}^{n+1} u_j^{(1)} \middle| \cdots \middle| \sum_{j=1}^{n+1} u_j^{(N)} \right)}. \tag{4.54}
\end{aligned}$$

Upon specialization $u_2^{(t)} = u_1^{(t)} + 1$, ($1 \leq t \leq N$), we have $\frac{[u_1^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N}]_r [u_1^{(t+1)} - u_j^{(t)} + \frac{s}{N}]_r}{[u_1^{(t)} - u_j^{(t)}]_r [u_j^{(t)} - u_1^{(t)} - 1]_r} = \frac{[u_i^{(t)} - u_2^{(t+1)} + 1 - \frac{s}{N}]_r [u_i^{(t+1)} - u_2^{(t)} + \frac{s}{N}]_r}{[u_i^{(t)} - u_2^{(t)}]_r [u_2^{(t)} - u_i^{(t)} - 1]_r}$. Hence we have $\text{Res}_{u_2^{(1)}=u_1^{(1)}+1} \cdots \text{Res}_{u_2^{(N)}=u_1^{(N)}+1} \text{LHS}(1, n) = \text{Res}_{u_2^{(1)}=u_1^{(1)}+1} \cdots \text{Res}_{u_2^{(N)}=u_1^{(N)}+1} \text{RHS}(1, n)$, using periodic condition $\widehat{\vartheta}_\alpha(u_1^{(1)} + k | \cdots | u_1^{(N)} + k) = \widehat{\vartheta}_\alpha(u_1^{(1)} | \cdots | u_1^{(N)})$. Both $\text{LHS}(1, n)$ and $\text{RHS}(1, n)$ are symmetric with respect to $u_1^{(t)}, u_2^{(t)}, \dots, u_{n+1}^{(t)}$, we have

$$\text{Res}_{u_{i_1}^{(1)}=u_{j_1}^{(1)}+1} \cdots \text{Res}_{u_{i_N}^{(N)}=u_{j_N}^{(N)}+1} \text{LHS}(1, n) = \text{Res}_{u_{i_1}^{(1)}=u_{j_1}^{(1)}+1} \cdots \text{Res}_{u_{i_N}^{(N)}=u_{j_N}^{(N)}+1} \text{RHS}(1, n), \tag{4.55}$$

for $1 \leq i_t \neq j_t \leq n+1$ and $1 \leq t \leq N$. After taking the residues finitely many times, every residue relation which comes from $\text{LHS}(1, n) = \text{RHS}(1, n)$, is reduced to the above

(4.55). Hence we have shown the starting relations $n > m = 1$. For the second, we show the general $n > m \geq 1$. We assume the relation $\text{LHS}(m-1, n-1) = \text{RHS}(m-1, n-1)$. Let us take the residue at $u_1^{(t)} = u_2^{(t)} + 1$, ($1 \leq t \leq N$). We have

$$\begin{aligned}
& \text{Res}_{u_2^{(1)}=u_1^{(1)}+1} \cdots \text{Res}_{u_2^{(N)}=u_1^{(N)}+1} (\text{LHS}(m, n) - \text{RHS}(m, n)) \\
&= \prod_{t=1}^N \text{Res}_{u_2^{(t)}=u_1^{(t)}+1} \frac{\left[u_1^{(t)} - u_2^{(t+1)} + 1 - \frac{s}{N} \right]_r \left[u_1^{(t+1)} - u_2^{(t)} + \frac{s}{N} \right]_r}{\left[u_1^{(t)} - u_2^{(t)} \right]_r \left[u_2^{(t)} - u_1^{(t)} - 1 \right]_r} \\
&\times \prod_{t=1}^N \prod_{j=3}^{m+n} \frac{\left[u_1^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \left[u_1^{(t+1)} - u_j^{(t)} + \frac{s}{N} \right]_r}{\left[u_1^{(t)} - u_j^{(t)} \right]_r \left[u_j^{(t)} - u_1^{(t)} - 1 \right]_r} \\
&\times \sum_{\substack{L_1 \cup L_1^c = \{3, 4, \dots, n+m\} \\ |L_1| = m-1, |L_1^c| = n-1}} \sum_{\substack{L_2 \cup L_2^c = \{3, 4, \dots, n+m\} \\ |L_2| = m-1, |L_2^c| = n-1}} \cdots \sum_{\substack{L_N \cup L_N^c = \{3, 4, \dots, n+m\} \\ |L_N| = m-1, |L_N^c| = n-1}} \\
&\times \frac{\widehat{\vartheta}_\alpha \left(\sum_{j \in L_1 \cup \{1\}} u_j^{(1)} \middle| \cdots \middle| \sum_{j \in L_N \cup \{1\}} u_j^{(N)} \right) \widehat{\vartheta}_\beta \left(\sum_{j \in L_1^c \cup \{1\}} u_j^{(1)} \middle| \cdots \middle| \sum_{j \in L_N^c \cup \{1\}} u_j^{(N)} \right)}{\widehat{\vartheta}_\gamma \left(\sum_{j=1}^{m+n} u_j^{(1)} \middle| \cdots \middle| \sum_{j=1}^{m+n} u_j^{(N)} \right)} \\
&\times \left(\prod_{t=1}^N \frac{\prod_{i \in L_t} \prod_{j \in L_{t+1}^c} \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i \in L_{t+1}} \prod_{j \in L_t^c} \left[u_i^{(t+1)} - u_j^{(t)} + \frac{s}{N} \right]_r}{\prod_{i \in L_t} \prod_{j \in L_{t+1}^c} \left[u_i^{(t)} - u_j^{(t)} \right]_r \left[u_j^{(t)} - u_i^{(t)} - 1 \right]_r} \right. \\
&\left. - \prod_{t=1}^N \frac{\prod_{i \in L_t^c} \prod_{j \in L_{t+1}} \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i \in L_{t+1}^c} \prod_{j \in L_t} \left[u_i^{(t+1)} - u_j^{(t)} + \frac{s}{N} \right]_r}{\prod_{i \in L_t^c} \prod_{j \in L_{t+1}} \left[u_i^{(t)} - u_j^{(t)} \right]_r \left[u_j^{(t)} - u_i^{(t)} - 1 \right]_r} \right) = 0.
\end{aligned} \tag{4.56}$$

We have already used the hypothesis for $(m-1, n-1)$. Both $\text{LHS}(m, n)$ and $\text{RHS}(m, n)$ are symmetric with respect to $u_1^{(t)}, u_2^{(t)}, \dots, u_{m+n}^{(t)}$, we have

$$\text{Res}_{u_{i_1}^{(1)}=u_{j_1}^{(1)}+1} \cdots \text{Res}_{u_{i_N}^{(N)}=u_{j_N}^{(N)}+1} \text{LHS}(m, n) = \text{Res}_{u_{i_1}^{(1)}=u_{j_1}^{(1)}+1} \cdots \text{Res}_{u_{i_N}^{(N)}=u_{j_N}^{(N)}+1} \text{RHS}(m, n), \tag{4.57}$$

for $1 \leq i_t \neq j_t \leq m+n$ and $1 \leq t \leq N$. After taking the residues finitely many times, every residue relation which comes from $\text{LHS}(m, n) = \text{RHS}(m, n)$, is reduced to the above (4.57). Hence we have shown $\text{LHS}(m, n) = \text{RHS}(m, n)$ for $n > m \geq 1$. Q.E.D.

Now let us show the commutation relation $[\mathcal{G}_m, \mathcal{G}_n] = 0$.

Proof of Theorem 4.2 We show $[\mathcal{G}_m, \mathcal{G}_n] = 0$ for $\text{Re}(r) > 0$ and $0 < \text{Re}(s) < N$. Others are shown by similar way. We use the integral representation of the nonlocal integrals of motion. In this regime, the integral contour exists in annulus. Hence we can use the notation $[\cdots]_{1, z_1 \cdots z_n}$. The following operators in the integrand of the nonlocal integrals of motion satisfies the S_n -invariance. For $\sigma_1, \sigma_2, \cdots, \sigma_N \in S_{m+n}$, we have

$$\begin{aligned}
& F_1(z_{\sigma_1(1)}^{(1)}) \cdots F_1(z_{\sigma_1(m+n)}^{(1)}) F_2(z_{\sigma_2(1)}^{(2)}) \cdots F_2(z_{\sigma_2(m+n)}^{(2)}) \cdots F_N(z_{\sigma_N(1)}^{(N)}) \cdots F_N(z_{\sigma_N(m+n)}^{(N)}) \\
& \times \prod_{t=1}^N \prod_{1 \leq i < j \leq m+n} \left[u_{\sigma_t(i)}^{(t)} - u_{\sigma_t(j)}^{(t)} \right]_r \left[u_{\sigma_t(j)}^{(t)} - u_{\sigma_t(i)}^{(t)} - 1 \right]_r \\
& = F_1(z_1^{(1)}) \cdots F_1(z_{m+n}^{(1)}) F_2(z_1^{(2)}) \cdots F_2(z_{m+n}^{(2)}) \cdots F_N(z_1^{(N)}) \cdots F_N(z_{m+n}^{(N)}) \\
& \times \prod_{t=1}^N \prod_{1 \leq i < j \leq m+n} \left[u_i^{(t)} - u_j^{(t)} \right]_r \left[u_j^{(t)} - u_i^{(t)} - 1 \right]_r. \tag{4.58}
\end{aligned}$$

Hence we have

$$\begin{aligned}
\mathcal{G}_m \cdot \mathcal{G}_n &= \left[F_1(z_1^{(1)}) \cdots F_1(z_{m+n}^{(1)}) F_2(z_1^{(2)}) \cdots F_2(z_{m+n}^{(2)}) \cdots F_N(z_1^{(N)}) \cdots F_N(z_{m+n}^{(N)}) \right. \\
& \times \frac{\prod_{t=1}^N \prod_{1 \leq i < j \leq m+n} \left[u_i^{(t)} - u_j^{(t)} \right]_r \left[u_j^{(t)} - u_i^{(t)} - 1 \right]_r}{\prod_{t=1}^{N-1} \prod_{i,j=1}^{m+n} \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r} \\
& \times \frac{1}{\prod_{i=1}^m \prod_{j=m+1}^{m+n} \left[u_i^{(N)} - u_j^{(1)} + \frac{s}{N} \right]_r \prod_{i=m+1}^{m+n} \prod_{j=1}^m \left[u_j^{(N)} - u_i^{(1)} - \frac{s}{N} + 1 \right]_r} \\
& \times \frac{1}{\prod_{i,j=1}^m \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r \prod_{i,j=m+1}^{m+n} \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r} \\
& \times \frac{1}{((m+n)!)^N} \sum_{\sigma_1 \in S_{m+n}} \cdots \sum_{\sigma_N \in S_{m+n}} \\
& \times \vartheta_\alpha \left(\sum_{j=1}^m u_{\sigma_1(j)}^{(1)} \middle| \cdots \middle| \sum_{j=1}^m u_{\sigma_N(j)}^{(N)} \right) \vartheta_\beta \left(\sum_{j=m+1}^{m+n} u_{\sigma_1(j)}^{(1)} \middle| \cdots \middle| \sum_{j=m+1}^{m+n} u_{\sigma_N(j)}^{(N)} \right)
\end{aligned}$$

$$\times \prod_{t=1}^N \prod_{i=1}^m \prod_{j=m+1}^{m+n} \frac{\left[u_{\sigma_t(i)}^{(t)} - u_{\sigma_{t+1}(j)}^{(t+1)} + 1 - \frac{s}{N} \right]_r \left[u_{\sigma_{t+1}(i)}^{(t+1)} - u_{\sigma_t(j)}^{(t)} + \frac{s}{N} \right]_r}{\left[u_{\sigma_t(i)}^{(t)} - u_{\sigma_t(j)}^{(t)} \right]_r \left[u_{\sigma_t(j)}^{(t)} - u_{\sigma_t(i)}^{(t)} - 1 \right]_r} \Bigg|_{1, z^{(1)}, \dots, z^{(N)}} \quad (4.59)$$

Therefore we have the following theta function identity as a sufficient condition of the commutation relations $\mathcal{G}_m \cdot \mathcal{G}_n = \mathcal{G}_n \cdot \mathcal{G}_m$.

$$\begin{aligned} & \sum_{\sigma_1 \in S_{m+n}} \sum_{\sigma_2 \in S_{m+n}} \cdots \sum_{\sigma_N \in S_{m+n}} \vartheta_\alpha \left(\sum_{j=1}^m u_{\sigma_1(j)}^{(1)} \middle| \sum_{j=1}^m u_{\sigma_2(j)}^{(2)} \middle| \cdots \middle| \sum_{j=1}^m u_{\sigma_N(j)}^{(N)} \right) \\ & \times \vartheta_\beta \left(\sum_{j=m+1}^{m+n} u_{\sigma_1(j)}^{(1)} \middle| \sum_{j=m+1}^{m+n} u_{\sigma_2(j)}^{(2)} \middle| \cdots \middle| \sum_{j=m+1}^{m+n} u_{\sigma_N(j)}^{(N)} \right) \\ & \times \prod_{t=1}^N \prod_{i=1}^m \prod_{j=m+1}^{m+n} \frac{\left[u_{\sigma_t(i)}^{(t)} - u_{\sigma_{t+1}(j)}^{(t+1)} - \frac{s}{N} \right]_r \left[u_{\sigma_{t+1}(i)}^{(t+1)} - u_{\sigma_t(j)}^{(t)} + \frac{s}{N} \right]_r}{\left[u_{\sigma_t(i)}^{(t)} - u_{\sigma_t(j)}^{(t)} \right]_r \left[u_{\sigma_t(j)}^{(t)} - u_{\sigma_t(i)}^{(t)} - 1 \right]_r} \\ & = \sum_{\sigma_1 \in S_{m+n}} \sum_{\sigma_2 \in S_{m+n}} \cdots \sum_{\sigma_N \in S_{m+n}} \vartheta_\beta \left(\sum_{j=1}^n u_{\sigma_1(j)}^{(1)} \middle| \sum_{j=1}^n u_{\sigma_2(j)}^{(2)} \middle| \cdots \middle| \sum_{j=1}^n u_{\sigma_N(j)}^{(N)} \right) \\ & \times \vartheta_\alpha \left(\sum_{j=n+1}^{m+n} u_{\sigma_1(j)}^{(1)} \middle| \sum_{j=n+1}^{m+n} u_{\sigma_2(j)}^{(2)} \middle| \cdots \middle| \sum_{j=n+1}^{m+n} u_{\sigma_N(j)}^{(N)} \right) \\ & \times \prod_{t=1}^N \prod_{i=1}^n \prod_{j=n+1}^{m+n} \frac{\left[u_{\sigma_t(i)}^{(t)} - u_{\sigma_{t+1}(j)}^{(t+1)} - \frac{s}{N} \right]_r \left[u_{\sigma_{t+1}(i)}^{(t+1)} - u_{\sigma_t(j)}^{(t)} + 1 - \frac{s}{N} \right]_r}{\left[u_{\sigma_t(i)}^{(t)} - u_{\sigma_t(j)}^{(t)} \right]_r \left[u_{\sigma_t(j)}^{(t)} - u_{\sigma_t(i)}^{(t)} - 1 \right]_r}. \quad (4.60) \end{aligned}$$

This is a special case $\nu_{\alpha,t} = \nu_{\beta,t} = 0$, ($1 \leq t \leq N$) of the theta identity in Proposition 4.5. Now we have shown Theorem. Q.E.D.

4.3 Proof of $[\mathcal{I}_m, \mathcal{G}_n] = 0$

In this section we give proof of the commutation relation $[\mathcal{I}_m, \mathcal{G}_n] = 0$. The fundamental operators $\Lambda_j(z)$ and $F_j(z)$ commute almost everywhere.

$$\begin{aligned} [\Lambda_j(z_1), F_j(z_2)] &= (-x^{r^*} + x^{-r^*}) \delta \left(x^{\frac{2s}{N}j-r} \frac{z_2}{z_1} \right) \mathcal{A}_j(x^{-r+\frac{2s}{N}j} z_1), \\ [\Lambda_{j+1}(z_1), F_j(z_2)] &= (x^{r^*} - x^{-r^*}) \delta \left(x^{\frac{2s}{N}j+r} \frac{z_2}{z_1} \right) \mathcal{A}_j(x^{r+\frac{2s}{N}j} z_1). \end{aligned}$$

Hence, in order to show the commutation relations, remaining task for us is to check whether delta-function factors cancell out or not. The Dynkin-automorphism invariance $\eta(\mathcal{I}_m) = \mathcal{I}_m$, $\eta(\mathcal{G}_m) = \mathcal{G}_m$, which we will show later, plays an important role in proof of this commutation relation $[\mathcal{I}_m, \mathcal{G}_n] = 0$.

Proof of Theorem 4.4 For a while we consider upon the regime: $0 < \text{Re}(s) < N$, $0 < \text{Re}(r) < 1$. At first we show the simple case, $[\mathcal{I}_1, \mathcal{G}_n] = 0$, for reader's convenience. Using Leibnitz-rule of adjoint action $[A, BC] = [A, B]C + B[A, C]$ and the invariance $\eta(\mathcal{I}_1) = \mathcal{I}_1$, we have

$$\begin{aligned}
[\mathcal{I}_1, \mathcal{G}_n] &= (x^{-r^*} - x^{r^*}) \sum_{t=1}^{N-1} \sum_{j=1}^n \prod_{u=1}^N \prod_{k=1}^n \int_{\tilde{C}(t,j)} \frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}} F_1(z_1^{(1)}) \cdots \\
&\times \cdots F_t(z_1^{(t)}) \cdots F_t(z_{j-1}^{(t)}) \mathcal{A}_t(x^{-r+\frac{2s}{N}t} z_j^{(t)}) F_t(z_{j+1}^{(t)}) \cdots F_t(z_n^{(t)}) \cdots F_N(z_n^{(N)}) \\
&\quad \prod_{t=1}^N \prod_{1 \leq i < j \leq n} \left[u_i^{(t)} - u_j^{(t)} \right]_r \left[u_j^{(t)} - u_i^{(t)} - 1 \right]_r \\
&\times \frac{\prod_{t=1}^{N-1} \prod_{i,j=1}^n \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^n \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r}{\prod_{t=1}^{N-1} \prod_{i,j=1}^n \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^n \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r} \\
&\times \vartheta \left(\sum_{j=1}^n u_j^{(1)} \middle| \sum_{j=1}^n u_j^{(2)} \middle| \cdots \middle| \sum_{j=1}^n u_j^{(N)} \right) \\
&+ (x^{r^*} - x^{-r^*}) \sum_{t=1}^{N-1} \sum_{j=1}^n \prod_{u=1}^N \prod_{k=1}^n \int_{\tilde{C}'(t,j)} \frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}} F_1(z_1^{(1)}) \cdots \\
&\times \cdots F_t(z_1^{(t)}) \cdots F_t(z_{j-1}^{(t)}) \mathcal{A}_t(x^{r+\frac{2s}{N}t} z_j^{(t)}) F_t(z_{j+1}^{(t)}) \cdots F_t(z_n^{(t)}) \cdots F_N(z_n^{(N)}) \\
&\quad \prod_{t=1}^N \prod_{1 \leq i < j \leq n} \left[u_i^{(t)} - u_j^{(t)} \right]_r \left[u_j^{(t)} - u_i^{(t)} - 1 \right]_r \\
&\times \frac{\prod_{t=1}^{N-1} \prod_{i,j=1}^n \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^n \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r}{\prod_{t=1}^{N-1} \prod_{i,j=1}^n \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^n \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r} \\
&\times \vartheta \left(\sum_{j=1}^n u_j^{(1)} \middle| \sum_{j=1}^n u_j^{(2)} \middle| \cdots \middle| \sum_{j=1}^n u_j^{(N)} \right) \\
&+ \eta \left(\sum_{j=1}^n \prod_{u=1}^N \prod_{k=1}^n \int_C \frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}} F_2(z_1^{(1)}) \cdots F_2(z_n^{(1)}) \cdots F_N(z_1^{(N-1)}) \cdots F_N(z_n^{(N-1)}) \right. \\
&\times F_1(z_1^{(N)}) \cdots F_1(z_{j-1}^{(N)}) [\mathcal{I}_1, F_1(z_j^{(N)})] F_1(z_{j+1}^{(N)}) \cdots F_1(z_n^{(N)}) \\
&\quad \prod_{t=1}^N \prod_{1 \leq i < j \leq n} \left[u_i^{(t)} - u_j^{(t)} \right]_r \left[u_j^{(t)} - u_i^{(t)} - 1 \right]_r \\
&\times \frac{\prod_{t=1}^{N-1} \prod_{i,j=1}^n \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^n \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r}{\prod_{t=1}^{N-1} \prod_{i,j=1}^n \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^n \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r}
\end{aligned}$$

$$\times \vartheta \left(\sum_{j=1}^n u_j^{(N)} \left| \sum_{j=1}^n u_j^{(1)} \right| \cdots \left| \sum_{j=1}^n u_j^{(N-1)} \right| \right). \quad (4.61)$$

Here we have set

$$\begin{aligned} \tilde{C}(t, j) &: |x^{4r-2+\frac{2s}{N}t} z_{j+1}^{(t)}|, \dots, |x^{4r-2+\frac{2s}{N}t} z_m^{(t)}| < |z_j^{(t)}| < |x^{-2r+2-\frac{2s}{N}t} z_1^{(t)}|, \dots, |x^{-2r+2-\frac{2s}{N}t} z_{j-1}^{(t)}|, \\ &|x^{2r+2-\frac{2s}{N}t} z_k^{(t-1)}| < |z_j^{(t)}| < |x^{2r-\frac{2s}{N}t} z_k^{(t-1)}|, \quad (1 \leq k \leq m), \\ &|x^{\frac{2s}{N}t} z_k^{(t+1)}| < |z_j^{(t)}| < |x^{-2+\frac{2s}{N}t} z_k^{(t+1)}|, \quad (1 \leq k \leq m), \end{aligned} \quad (4.62)$$

$$\begin{aligned} \tilde{C}'(t, j) &: |x^{2r-2+\frac{2s}{N}t} z_{j+1}^{(t)}|, \dots, |x^{2r-2+\frac{2s}{N}t} z_m^{(t)}| < |z_j^{(t)}| < |x^{-4r+2-\frac{2s}{N}t} z_1^{(t)}|, \dots, |x^{-4r+2-\frac{2s}{N}t} z_{j-1}^{(t)}|, \\ &|x^{2-\frac{2s}{N}t} z_k^{(t-1)}| < |z_j^{(t)}| < |x^{-\frac{2s}{N}t} z_k^{(t-1)}|, \quad (1 \leq k \leq m), \\ &|x^{-2r+\frac{2s}{N}t} z_k^{(t+1)}| < |z_j^{(t)}| < |x^{-2r-2+\frac{2s}{N}t} z_k^{(t+1)}|, \quad (1 \leq k \leq m). \end{aligned} \quad (4.63)$$

Let us change the variable $z_j^{(t)} \rightarrow x^{2r} z_j^{(t)}$ in the first term of (4.61). Using the periodicity of the integrand, we deform the first term to the second term of (4.61).

$$\begin{aligned} & \prod_{u=1}^N \prod_{k=1}^n \int_{\tilde{I}(t,j)} \frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}} F_1(z_1^{(1)}) \cdots F_t(z_1^{(t)}) \cdots F_t(z_{j-1}^{(t)}) \\ & \times \mathcal{A}_t(x^{-r} z_j^{(t)}) F_t(z_{j+1}^{(t)}) \cdots F_t(z_n^{(t)}) \cdots F_N(z_n^{(N)}) \\ & \times \frac{\prod_{t=1}^N \prod_{1 \leq i < j \leq n} [u_i^{(t)} - u_j^{(t)}]_r [u_j^{(t)} - u_i^{(t)} - 1]_r}{\prod_{t=1}^{N-1} \prod_{i,j=1}^n \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^n \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r} \vartheta \left(\sum_{j=1}^n u_j^{(1)} \left| \sum_{j=1}^n u_j^{(2)} \right| \cdots \left| \sum_{j=1}^n u_j^{(N)} \right| \right) \\ & = \prod_{u=1}^N \prod_{k=1}^n \int_{\tilde{I}(t,j)} \frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}} F_1(z_1^{(1)}) \cdots F_t(z_1^{(t)}) \cdots F_t(z_{j-1}^{(t)}) \\ & \times \mathcal{A}_t(x^r z_j^{(t)}) F_t(z_{j+1}^{(t)}) \cdots F_t(z_n^{(t)}) \cdots F_N(z_n^{(N)}) \\ & \times \frac{\prod_{t=1}^N \prod_{1 \leq i < j \leq n} [u_i^{(t)} - u_j^{(t)}]_r [u_j^{(t)} - u_i^{(t)} - 1]_r}{\prod_{t=1}^{N-1} \prod_{i,j=1}^n \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^n \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r} \vartheta \left(\sum_{j=1}^n u_j^{(1)} \left| \sum_{j=1}^n u_j^{(2)} \right| \cdots \left| \sum_{j=1}^n u_j^{(N)} \right| \right). \end{aligned} \quad (4.64)$$

Hence we have

$$\eta([\mathcal{I}_1, \mathcal{G}_n])$$

$$\begin{aligned}
&= (x^{-r^*} - x^{r^*}) \sum_{j=1}^n \prod_{u=1}^N \prod_{k=1}^n \int_{C(N,j)} \frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}} F_2(z_1^{(1)}) \cdots F_2(z_n^{(1)}) \cdots F_N(z_1^{(N-1)}) \cdots F_N(z_n^{(N-1)}) \\
&\times F_1(z_1^{(N)}) \cdots F_1(z_{j-1}^{(N)}) \mathcal{A}_1(x^{-r+\frac{2s}{N}} z_j^{(N)}) F_1(z_{j+1}^{(N)}) \cdots F_1(z_n^{(N)}) \\
&\quad \prod_{t=1}^N \prod_{1 \leq i < j \leq n} \left[u_i^{(t)} - u_j^{(t)} \right]_r \left[u_j^{(t)} - u_i^{(t)} - 1 \right]_r \\
&\times \frac{\prod_{t=1}^{N-1} \prod_{i,j=1}^n \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^n \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r}{\prod_{t=1}^{N-1} \prod_{i,j=1}^n \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^n \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r} \\
&\times \vartheta \left(\sum_{j=1}^n u_j^{(1)} \middle| \sum_{j=1}^n u_j^{(2)} \cdots \sum_{j=1}^n u_j^{(N)} \right) \\
&- (x^{-r^*} - x^{r^*}) \sum_{j=1}^n \prod_{u=1}^N \prod_{k=1}^n \int_{C'(N,j)} \frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}} F_2(z_1^{(1)}) \cdots F_2(z_n^{(1)}) \cdots F_N(z_1^{(N-1)}) \cdots F_N(z_n^{(N-1)}) \\
&\times F_1(z_1^{(N)}) \cdots F_1(z_{j-1}^{(N)}) \mathcal{A}_1(x^{r+\frac{2s}{N}} z_j^{(N)}) F_1(z_{j+1}^{(N)}) \cdots F_1(z_n^{(N)}) \\
&\quad \prod_{t=1}^N \prod_{1 \leq i < j \leq n} \left[u_i^{(t)} - u_j^{(t)} \right]_r \left[u_j^{(t)} - u_i^{(t)} - 1 \right]_r \\
&\times \frac{\prod_{t=1}^{N-1} \prod_{i,j=1}^n \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^n \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r}{\prod_{t=1}^{N-1} \prod_{i,j=1}^n \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^n \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r} \\
&\times \vartheta \left(\sum_{j=1}^n u_j^{(N)} \middle| \sum_{j=1}^n u_j^{(1)} \cdots \sum_{j=1}^n u_j^{(N-1)} \right). \tag{4.65}
\end{aligned}$$

Here we have set

$$\begin{aligned}
\tilde{C}(N, j) &: |x^{4r-2+\frac{2s}{N}} z_{j+1}^{(N)}|, \dots, |x^{4r-2+\frac{2s}{N}} z_m^{(N)}| < |z_j^{(N)}| < |x^{-2r+2-\frac{2s}{N}} z_1^{(N)}|, \dots, |x^{-2r+2-\frac{2s}{N}} z_{j-1}^{(N)}|, \\
&|x^{2r+2-\frac{2s}{N}} z_k^{(N-1)}| < |z_j^{(N)}| < |x^{2r-\frac{2s}{N}} z_k^{(N-1)}|, \quad (1 \leq k \leq m), \\
&|x^{\frac{2s}{N}} z_k^{(1)}| < |z_j^{(N)}| < |x^{-2+\frac{2s}{N}} z_k^{(1)}|, \quad (1 \leq k \leq m), \tag{4.66}
\end{aligned}$$

$$\begin{aligned}
\tilde{C}'(N, j) &: |x^{2r-2+\frac{2s}{N}} z_{j+1}^{(N-1)}|, \dots, |x^{2r-2+\frac{2s}{N}} z_m^{(N-1)}| \\
&< |z_j^{(N)}| < |x^{-4r+2-\frac{2s}{N}} z_1^{(N-1)}|, \dots, |x^{-4r+2-\frac{2s}{N}} z_{j-1}^{(N-1)}|, \\
&|x^{2-\frac{2s}{N}} z_k^{(N-1)}| < |z_j^{(N)}| < |x^{-\frac{2s}{N}} z_k^{(N-1)}|, \quad (1 \leq k \leq m), \\
&|x^{-2r+\frac{2s}{N}} z_k^{(1)}| < |z_j^{(N)}| < |x^{-2r-2+\frac{2s}{N}} z_k^{(1)}|, \quad (1 \leq k \leq m). \tag{4.67}
\end{aligned}$$

Using the periodicity of the integrand, we have $[\mathcal{I}_1, \mathcal{G}_n] = 0$. For the second, we consider the commutation relation $[\mathcal{I}_m, \mathcal{G}_n] = 0$. By using Leibnitz-rule of adjoint action and the invariance $\eta(\mathcal{I}_m) = \mathcal{I}_m$, we have

$$[\mathcal{I}_m, \mathcal{G}_n]$$

$$\begin{aligned}
&= (x^{-r^*} - x^{r^*}) \sum_{t=1}^{N-1} \sum_{i=1}^m \sum_{j=1}^n \prod_{u=1}^N \prod_{k=1}^n \int_{\tilde{C}(t,j)} \frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}} \vartheta \left(\sum_{j=1}^n u_j^{(1)} \left| \sum_{j=1}^n u_j^{(2)} \right| \cdots \left| \sum_{j=1}^n u_j^{(N)} \right| \right) \\
&\quad \prod_{t=1}^N \prod_{1 \leq i < j \leq n} \left[u_i^{(t)} - u_j^{(t)} \right]_r \left[u_j^{(t)} - u_i^{(t)} - 1 \right]_r \\
&\times \frac{\prod_{t=1}^{N-1} \prod_{i,j=1}^n \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^n \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r}{\prod_{t=1}^{N-1} \prod_{i,j=1}^n \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^n \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r} \\
&\times \left(\prod_{k=1}^m \int_{I \cap \left\{ \left| \frac{x^{-r+\frac{2s}{N}t} z_j^{(t)}}{w_i} \right| < 1 \right\}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{1}{\left(1 - \frac{x^{-r+\frac{2s}{N}t} z_j^{(t)}}{w_i} \right)} \right. \\
&\quad \left. + \prod_{k=1}^m \int_{I \cap \left\{ \left| \frac{x^{-r+\frac{2s}{N}t} z_j^{(t)}}{w_i} \right| > 1 \right\}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\frac{x^{r-\frac{2s}{N}t} w_i}{z_j^{(t)}}}{\left(1 - \frac{x^{r-\frac{2s}{N}t} w_i}{z_j^{(t)}} \right)} \right) \\
&\times \prod_{1 \leq k < l \leq m} h(v_k - v_l) T_1(w_1) \cdots T_1(w_{i-1}) F_1(z_1^{(t)}) \cdots F_t(z_{j-1}^{(t)}) \\
&\times \mathcal{A}_t(x^{-r+\frac{2s}{N}t} z_j^{(t)}) F_t(z_{j+1}^{(t)}) \cdots F_N(z_n^{(t)}) T_1(w_{i+1}) \cdots T_1(w_m) \\
&- (x^{-r^*} - x^{r^*}) \sum_{t=1}^{N-1} \sum_{i=1}^m \sum_{j=1}^n \prod_{u=1}^N \prod_{k=1}^n \int_{\tilde{C}'(t,j)} \frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}} \vartheta \left(\sum_{j=1}^n u_j^{(1)} \left| \sum_{j=1}^n u_j^{(2)} \right| \cdots \left| \sum_{j=1}^n u_j^{(N)} \right| \right) \\
&\quad \prod_{t=1}^N \prod_{1 \leq i < j \leq n} \left[u_i^{(t)} - u_j^{(t)} \right]_r \left[u_j^{(t)} - u_i^{(t)} - 1 \right]_r \\
&\times \frac{\prod_{t=1}^{N-1} \prod_{i,j=1}^n \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^n \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r}{\prod_{t=1}^{N-1} \prod_{i,j=1}^n \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^n \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r} \\
&\times \left(\prod_{k=1}^m \int_{I \cap \left\{ \left| \frac{x^{r+\frac{2s}{N}t} z_j^{(t)}}{w_i} \right| < 1 \right\}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{1}{\left(1 - \frac{x^{r+\frac{2s}{N}t} z_j^{(t)}}{w_i} \right)} \right. \\
&\quad \left. + \prod_{k=1}^m \int_{I \cap \left\{ \left| \frac{x^{r+\frac{2s}{N}t} z_j^{(t)}}{w_i} \right| > 1 \right\}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\frac{x^{-r-\frac{2s}{N}t} w_i}{z_j^{(t)}}}{\left(1 - \frac{x^{-r-\frac{2s}{N}t} w_i}{z_j^{(t)}} \right)} \right) \\
&\times \prod_{1 \leq k < l \leq m} h(v_k - v_l) T_1(w_1) \cdots T_1(w_{i-1}) F_1(z_1^{(t)}) \cdots F_t(z_{j-1}^{(t)}) \\
&\times \mathcal{A}_t(x^{r+\frac{2s}{N}t} z_j^{(t)}) F_t(z_{j+1}^{(t)}) \cdots F_N(z_n^{(t)}) T_1(w_{i+1}) \cdots T_1(w_m)
\end{aligned}$$

$$\begin{aligned}
& + \eta \left(\sum_{i=1}^m \sum_{j=1}^n \prod_{u=1}^N \prod_{k=1}^n \int_C \frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}} \vartheta \left(\sum_{j=1}^n u_j^{(N)} \middle| \sum_{j=1}^n u_j^{(1)} \middle| \cdots \middle| \sum_{j=1}^n u_j^{(N-1)} \right) \right. \\
& \quad \prod_{t=1}^N \prod_{1 \leq i < j \leq m} \left[u_i^{(t)} - u_j^{(t)} \right]_r \left[u_j^{(t)} - u_i^{(t)} - 1 \right]_r \\
& \times \frac{\prod_{t=1}^{N-1} \prod_{i,j=1}^n \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^n \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r}{\prod_{k=1}^m \int_I \frac{dw_k}{2\pi\sqrt{-1}w_k} \prod_{1 \leq k < l \leq m} h(v_k - v_l) T_1(w_1) \cdots T_1(w_{i-1})} \\
& \times F_2(z_1^{(1)}) \cdots F_2(z_n^{(1)}) \cdots F_N(z_1^{(N-1)}) \cdots F_N(z_n^{(N-1)}) F_1(z_1^{(N)}) \cdots F_1(z_{j-1}^{(N)}) \\
& \times [T_1(w_i), F_1(z_j^{(N)})] F_1(z_{j+1}^{(N)}) \cdots F_1(z_n^{(N)}) T_1(w_{i+1}) \cdots T_1(w_m). \tag{4.68}
\end{aligned}$$

Here $\tilde{C}(t, j)$, $\tilde{C}'(t, j)$ are given as the same manner in proof of $[\mathcal{I}_1, \mathcal{G}_n] = 0$. Let us change the variable $z_j^{(t)} \rightarrow x^{2r} z_j^{(t)}$ in the first term, upon the conditions $0 < \text{Re}(s) < N$, $0 < \text{Re}(r) < 1$. Using periodicity of the integrands, we have the cancellation of the first and the second terms of (4.68).

$$\begin{aligned}
& \prod_{u=1}^N \prod_{k=1}^n \int_{\tilde{C}(t,j)} \frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}} \vartheta \left(\sum_{j=1}^n u_j^{(1)} \middle| \sum_{j=1}^n u_j^{(2)} \middle| \cdots \middle| \sum_{j=1}^n u_j^{(N)} \right) \\
& \quad \prod_{t=1}^N \prod_{1 \leq i < j \leq n} \left[u_i^{(t)} - u_j^{(t)} \right]_r \left[u_j^{(t)} - u_i^{(t)} - 1 \right]_r \\
& \times \frac{\prod_{t=1}^{N-1} \prod_{i,j=1}^n \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^n \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r}{\prod_{k=1}^m \int_{I \cap \left\{ \left| \frac{x^{-r+\frac{2s}{N}t} z_j^{(t)}}{w_i} \right| < 1 \right\}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{1}{\left(1 - \frac{x^{-r+\frac{2s}{N}t} z_j^{(t)}}{w_i} \right)}} \\
& \quad + \prod_{k=1}^m \int_{I \cap \left\{ \left| \frac{x^{-r+\frac{2s}{N}t} z_j^{(t)}}{w_i} \right| > 1 \right\}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\frac{x^{r-\frac{2s}{N}t} w_i}{z_j^{(t)}}}{\left(1 - \frac{x^{r-\frac{2s}{N}t} w_i}{z_j^{(t)}} \right)} \\
& \times \prod_{1 \leq k < l \leq m} h(v_k - v_l) T_1(w_1) \cdots T_1(w_{i-1}) F_1(z_1^{(t)}) \cdots F_t(z_{j-1}^{(t)}) \\
& \times \mathcal{A}_t(x^{-r+\frac{2s}{N}t} z_j^{(t)}) F_t(z_{j+1}^{(t)}) \cdots F_N(z_n^{(t)}) T_1(w_{i+1}) \cdots T_1(w_m) \\
& = \prod_{u=1}^N \prod_{k=1}^n \int_{\tilde{C}(t,j)} \frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}} \vartheta \left(\sum_{j=1}^m u_j^{(1)} \middle| \sum_{j=1}^n u_j^{(2)} \middle| \cdots \middle| \sum_{j=1}^n u_j^{(N)} \right)
\end{aligned}$$

$$\begin{aligned}
& \prod_{t=1}^N \prod_{1 \leq i < j \leq m} \left[u_i^{(t)} - u_j^{(t)} \right]_r \left[u_j^{(t)} - u_i^{(t)} - 1 \right]_r \\
& \times \frac{\prod_{t=1}^{N-1} \prod_{i,j=1}^n \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^n \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r}{\prod_{t=1}^{N-1} \prod_{i,j=1}^n \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^n \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r} \\
& \times \left(\prod_{k=1}^m \int_{I \cap \left\{ \left| \frac{x^{r+\frac{2s}{N}t} z_j^{(t)}}{w_i} \right| < 1 \right\}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{1}{\left(1 - \frac{x^{r+\frac{2s}{N}t} z_j^{(t)}}{w_i} \right)} \right. \\
& \quad \left. + \prod_{k=1}^m \int_{I \cap \left\{ \left| \frac{x^{r+\frac{2s}{N}t} z_j^{(t)}}{w_i} \right| > 1 \right\}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\frac{x^{-r-\frac{2s}{N}t} w_i}{z_j^{(t)}}}{\left(1 - \frac{x^{-r-\frac{2s}{N}t} w_i}{z_j^{(t)}} \right)} \right) \\
& \times \prod_{1 \leq k < l \leq m} h(v_k - v_l) T_1(w_1) \cdots T_1(w_{i-1}) F_1(z_1^{(t)}) \cdots F_t(z_{j-1}^{(t)}) \\
& \times \mathcal{A}_t(x^{r+\frac{2s}{N}t} z_j^{(t)}) F_t(z_{j+1}^{(t)}) \cdots F_N(z_n^{(t)}) T_1(w_{i+1}) \cdots T_1(w_m). \tag{4.69}
\end{aligned}$$

Hence we have

$$\begin{aligned}
& \eta([\mathcal{I}_m, \mathcal{G}_n]) \\
& = (x^{-r^*} - x^{r^*}) \sum_{i=1}^m \sum_{j=1}^n \prod_{u=1}^N \prod_{k=1}^n \int_{\tilde{C}(N,j)} \frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}} \vartheta \left(\sum_{j=1}^n u_j^{(N)} \left| \sum_{j=1}^n u_j^{(1)} \right| \cdots \left| \sum_{j=1}^n u_j^{(N-1)} \right| \right) \\
& \quad \prod_{t=1}^N \prod_{1 \leq i < j \leq n} \left[u_i^{(t)} - u_j^{(t)} \right]_r \left[u_j^{(t)} - u_i^{(t)} - 1 \right]_r \\
& \times \frac{\prod_{t=1}^{N-1} \prod_{i,j=1}^n \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^n \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r}{\prod_{t=1}^{N-1} \prod_{i,j=1}^n \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^n \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r} \\
& \times \left(\prod_{k=1}^m \int_{I \cap \left\{ \left| \frac{x^{-r+\frac{2s}{N}} z_j^{(N)}}{w_i} \right| < 1 \right\}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{1}{\left(1 - \frac{x^{-r+\frac{2s}{N}} z_j^{(N)}}{w_i} \right)} \right. \\
& \quad \left. + \prod_{k=1}^m \int_{I \cap \left\{ \left| \frac{x^{-r+\frac{2s}{N}} z_j^{(N)}}{w_i} \right| > 1 \right\}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\frac{x^{r-\frac{2s}{N}} w_i}{z_j^{(N)}}}{\left(1 - \frac{x^{r-\frac{2s}{N}} w_i}{z_j^{(N)}} \right)} \right) \\
& \times \prod_{1 \leq k < l \leq m} h(v_k - v_l) T_1(w_1) \cdots T_1(w_{i-1}) \\
& \times F_2(z_1^{(1)}) \cdots F_2(z_n^{(1)}) \cdots F_N(z_1^{(N-1)}) \cdots F_N(z_n^{(N-1)}) F_1(z_1^{(N)}) \cdots F_1(z_{j-1}^{(N)}) \\
& \times \mathcal{A}_1(x^{-r+\frac{2s}{N}} z_j^{(N)}) F_1(z_{j+1}^{(N)}) \cdots F_1(z_n^{(N)}) T_1(w_{i+1}) \cdots T_1(w_m)
\end{aligned}$$

$$\begin{aligned}
& - (x^{-r^*} - x^{r^*}) \sum_{i=1}^m \sum_{j=1}^n \prod_{u=1}^N \prod_{k=1}^n \int_{\tilde{C}'(N,j)} \frac{dz_k^{(u)}}{2\pi\sqrt{-1}z_k^{(u)}} \vartheta \left(\sum_{j=1}^n u_j^{(N)} \middle| \sum_{j=1}^n u_j^{(1)} \middle| \cdots \middle| \sum_{j=1}^n u_j^{(N-1)} \right) \\
& \quad \prod_{t=1}^N \prod_{1 \leq i < j \leq n} \left[u_i^{(t)} - u_j^{(t)} \right]_r \left[u_j^{(t)} - u_i^{(t)} - 1 \right]_r \\
& \times \frac{\prod_{t=1}^{N-1} \prod_{i,j=1}^n \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{s}{N} \right]_r \prod_{i,j=1}^n \left[u_i^{(1)} - u_j^{(N)} + \frac{s}{N} \right]_r}{\prod_{k=1}^m \int_{I \cap \left\{ \left| \frac{x^{r+\frac{2s}{N}} z_j^{(N)}}{w_i} \right| < 1 \right\}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{1}{\left(1 - \frac{x^{r+\frac{2s}{N}} z_j^{(N)}}{w_i} \right)}} \\
& \quad + \prod_{k=1}^m \int_{I \cap \left\{ \left| \frac{x^{r+\frac{2s}{N}} z_j^{(N)}}{w_i} \right| > 1 \right\}} \frac{dw_k}{2\pi\sqrt{-1}w_k} \frac{\frac{x^{-r-\frac{2s}{N}} w_i}{z_j^{(N)}}}{\left(1 - \frac{x^{-r-\frac{2s}{N}} w_i}{z_j^{(N)}} \right)} \\
& \times \prod_{1 \leq k < l \leq m} h(v_k - v_l) T_1(w_1) \cdots T_1(w_{i-1}) \\
& \times F_2(z_1^{(1)}) \cdots F_2(z_n^{(1)}) \cdots F_N(z_1^{(N-1)}) \cdots F_N(z_n^{(N-1)}) F_1(z_1^{(N)}) \cdots F_1(z_{j-1}^{(N)}) \\
& \times \mathcal{A}_1(x^{r+\frac{2s}{N}} z_j^{(N)}) F_1(z_{j+1}^{(N)}) \cdots F_1(z_n^{(N)}) T_1(w_{i+1}) \cdots T_1(w_m), \tag{4.70}
\end{aligned}$$

where $\tilde{C}(N, j)$, $\tilde{C}'(N, j)$ are given as the same manner as the case of $[\mathcal{I}_1, \mathcal{G}_n] = 0$. Changing variable $z_j^{(N)} \rightarrow x^{2r} z_j^{(N)}$, we have the commutation relation $[\mathcal{I}_m, \mathcal{G}_n] = 0$. Other commutation relations : $[\mathcal{I}_m, \mathcal{G}_n^*] = [\mathcal{I}_m^*, \mathcal{G}_n] = [\mathcal{I}_m^*, \mathcal{G}_n^*] = 0$ are shown by similar way. We omit details. Q.E.D.

4.4 Proof of $[\mathcal{G}_m, \mathcal{G}_n^*] = 0$

In this section we give proof of the commutation relation $[\mathcal{I}_m, \mathcal{G}_n] = 0$. The fundamental operators $E_j(z)$ and $F_j(z)$ commute almost everywhere.

$$[E_j(z_1), F_j(z_2)] = \frac{1}{x - x^{-1}} (\delta(xz_2/z_1) H_j(x^r z_2) - \delta(xz_1/z_2) H_j(x^{-r} z_2)).$$

Hence, in order to show the commutation relations, remaining task for us is to check whether delta-function factors cancell out or not.

Proof of Theorem 4.3 We consider the regime $\text{Re}(r) > 0$ and $\text{Re}(r^*) < 0$. At first we consider the simple case : $[\mathcal{G}_1, \mathcal{G}_1^*] = 0$. Using Leibnitz-rule of adjoint action and the

commutation relations of screening currents $E_j(z), F_j(z)$, we have

$$\begin{aligned}
[\mathcal{G}_1^*, \mathcal{G}_1] &= \sum_{t=1}^N \prod_{q=1}^N \prod_{\substack{p=1 \\ p \neq t}}^N \oint_{C^{(t)}} \frac{dz^{(q)}}{2\pi\sqrt{-1}z^{(q)}} \frac{dw^{(p)}}{2\pi\sqrt{-1}w^{(p)}} B^{(t)}(x^r z^{(t)}; \{z^{(q)}\}_{1 \leq q \leq N}, \{w^{(q)}\}_{\substack{1 \leq q \leq N \\ q \neq t}}) \\
&- \sum_{t=1}^N \prod_{q=1}^N \prod_{\substack{p=1 \\ p \neq t}}^N \oint_{\tilde{C}^{(t)}} \frac{dz^{(q)}}{2\pi\sqrt{-1}z^{(q)}} \frac{dw^{(p)}}{2\pi\sqrt{-1}w^{(p)}} B^{(t)}(x^{-r} z^{(t)}; \{z^{(q)}\}_{1 \leq q \leq N}, \{w^{(q)}\}_{\substack{1 \leq q \leq N \\ q \neq t}}).
\end{aligned} \tag{4.71}$$

Here we have set

$$\begin{aligned}
&B^{(t)}(z; \{z^{(q)}\}_{1 \leq q \leq N}, \{w^{(q)}\}_{\substack{1 \leq q \leq N \\ q \neq t}}) \\
&= \frac{1}{x - x^{-1}} F_1(z^{(1)}) \cdots F_{t-1}(z^{(t-1)}) E_1(w^{(1)}) \cdots E_{t-1}(w^{(t-1)}) \\
&\times H_t(z) E_{t+1}(w^{(t+1)}) \cdots E_N(w^{(N)}) F_{t+1}(z^{(t+1)}) \cdots F_N(z^{(N)}) \\
&\times \frac{\vartheta(u^{(1)} | \cdots | u^{(N)})}{\prod_{q=1}^{N-1} [u^{(q)} - u^{(q+1)} + 1 - \frac{s}{N}]_r [u^{(1)} - u^{(N)} + \frac{s}{N}]_r} \\
&\times \frac{\vartheta^*(v^{(1)} | \cdots | v^{(N)})}{\prod_{q=1}^{N-1} [v^{(q)} - v^{(q+1)} + 1 - \frac{s}{N}]_{-r^*} [v^{(1)} - v^{(N)} + \frac{s}{N}]_{-r^*}} \Bigg|_{v^{(t)} = u - \frac{r^*}{2}}
\end{aligned} \tag{4.72}$$

Here the contours $C^{(t)}$ and $\tilde{C}^{(t)}$ are characterized by

$$\begin{aligned}
C^{(t)} : & |x^{2-\frac{2s}{N}} z^{(t-1)}|, |x^{\frac{2s}{N}} z^{(t+1)}| < |z^{(t)}| < |x^{-2r-\frac{2s}{N}} z^{(t-1)}|, |x^{-2-2r+\frac{2s}{N}} z^{(t+1)}|, \\
& |x^{-2r+3-\frac{2s}{N}} w^{(t-1)}|, |x^{-2r+1+\frac{2s}{N}} w^{(t+1)}| < |z^{(t)}| < |x^{-1-\frac{2s}{N}} w^{(t-1)}|, |x^{-3+\frac{2s}{N}} w^{(t+1)}|, \\
& |x^{\frac{2s}{N}} z^{(q+1)}| < |z^{(q)}| < |x^{-2+\frac{2s}{N}} z^{(q+1)}|, \quad (1 \leq q(\neq t, t-1) \leq N), \\
& |x^{\frac{2s}{N}} w^{(q+1)}| < |w^{(q)}| < |x^{-2+\frac{2s}{N}} w^{(q+1)}|, \quad (1 \leq q(\neq t, t-1) \leq N),
\end{aligned} \tag{4.73}$$

$$\begin{aligned}
\tilde{C}^{(t)} : & |x^{2r+2-\frac{2s}{N}} z^{(t-1)}|, |x^{2r+\frac{2s}{N}} z^{(t+1)}| < |z^{(t)}| < |x^{-\frac{2s}{N}} z^{(t-1)}|, |x^{-2+\frac{2s}{N}} z^{(t+1)}|, \\
& |x^{3-\frac{2s}{N}} w^{(t-1)}|, |x^{1+\frac{2s}{N}} w^{(t+1)}| < |z^{(t)}| < |x^{2r-1-\frac{2s}{N}} w^{(t-1)}|, |x^{2r-3+\frac{2s}{N}} w^{(t+1)}|, \\
& |x^{\frac{2s}{N}} z^{(q+1)}| < |z^{(q)}| < |x^{-2+\frac{2s}{N}} z^{(q+1)}|, \quad (1 \leq q(\neq t, t-1) \leq N), \\
& |x^{\frac{2s}{N}} w^{(q+1)}| < |w^{(q)}| < |x^{-2+\frac{2s}{N}} w^{(q+1)}|, \quad (1 \leq q(\neq t, t-1) \leq N).
\end{aligned} \tag{4.74}$$

Let us change the variable $z^{(t)} \rightarrow x^{2r} z^{(t)}$ of the integrand $B^{(t)}(x^{-r} z^{(t)}; \{z^{(q)}\}_{1 \leq q \leq N}, \{w^{(q)}\}_{\substack{1 \leq q \leq N \\ q \neq t}})$ and the contour $C^{(t)}$. Using the periodic condition of theta function $[u+r]_r = -[u]_r$, we

have $B^{(t)}(x^r z^{(t)}; \{z^{(q)}\}_{1 \leq q \leq N}, \{w^{(q)}\}_{1 \leq q \leq N, q \neq t})$ and $\tilde{C}^{(t)}$. Hence we have

$$\begin{aligned} & \prod_{q=1}^N \prod_{\substack{p=1 \\ p \neq t}}^N \oint_{C^{(t)}} \frac{dz^{(q)}}{2\pi\sqrt{-1}z^{(q)}} \frac{dw^{(p)}}{2\pi\sqrt{-1}w^{(p)}} B^{(t)}(x^r z^{(t)}; \{z^{(q)}\}_{1 \leq q \leq N}, \{w^{(q)}\}_{1 \leq q \leq N, q \neq t}) \\ &= \prod_{q=1}^N \prod_{\substack{p=1 \\ p \neq t}}^N \oint_{\tilde{C}^{(t)}} \frac{dz^{(q)}}{2\pi\sqrt{-1}z^{(q)}} \frac{dw^{(p)}}{2\pi\sqrt{-1}w^{(p)}} B^{(t)}(x^{-r} z^{(t)}; \{z^{(q)}\}_{1 \leq q \leq N}, \{w^{(q)}\}_{1 \leq q \leq N, q \neq t}) \end{aligned} \quad (4.75)$$

Therefore we have $[\mathcal{G}_1^*, \mathcal{G}_1] = 0$. Generalization to generic parameter $0 < \text{Re}(r) < 1$ and $s \in \mathbb{C}$ should be understood as analytic continuation. For the second, we consider $[\mathcal{G}_m^*, \mathcal{G}_n] = 0$. Using Leibnitz-rule of adjoint action and the commutation relations of screening currents $E_j(z), F_j(z)$, we have

$$\begin{aligned} [\mathcal{G}_m^*, \mathcal{G}_n] &= \sum_{t=1}^N \sum_{i=1}^n \sum_{j=1}^m \prod_{q=1}^N \prod_{k=1}^n \prod_{p=1}^N \prod_{\substack{l=1 \\ l \neq t}}^m \oint_{C_{i,j}^{(t)}} \frac{dz^{(q)k}}{2\pi\sqrt{-1}z_k^{(q)}} \frac{dw^{(p)l}}{2\pi\sqrt{-1}w_l^{(p)}} \\ &\quad \times B_{i,j}^{(t)} \left(x^{-r} z_i^{(t)}; \{z_j^{(q)}\}_{\substack{1 \leq q \leq N \\ 1 \leq k \leq n}}, \{w^{(q)}\}_{\substack{1 \leq q \leq N \\ 1 \leq l \neq j \leq m}} \right) \\ &\quad - \sum_{t=1}^N \sum_{i=1}^n \sum_{j=1}^m \prod_{q=1}^N \prod_{k=1}^n \prod_{p=1}^N \prod_{\substack{l=1 \\ l \neq t}}^m \oint_{C_{i,j}^{(t)}} \frac{dz^{(q)k}}{2\pi\sqrt{-1}z_k^{(q)}} \frac{dw^{(p)l}}{2\pi\sqrt{-1}w_l^{(p)}} \\ &\quad \times B_{i,j}^{(t)} \left(x^r z_i^{(t)}; \{z_j^{(q)}\}_{\substack{1 \leq q \leq N \\ 1 \leq k \leq n}}, \{w^{(q)}\}_{\substack{1 \leq q \leq N \\ 1 \leq l \neq j \leq m}} \right). \end{aligned} \quad (4.76)$$

Here we have set

$$\begin{aligned} & B_{i,j}^{(t)} \left(z; \{z_j^{(q)}\}_{\substack{1 \leq q \leq N \\ 1 \leq k \leq n}}, \{w^{(q)}\}_{\substack{1 \leq q \leq N \\ 1 \leq l \neq j \leq m}} \right) \\ &= \frac{1}{x - x^{-1}} F_1(z_1^{(1)}) \cdots F_1(z_n^{(1)}) E_1(w_1^{(1)}) \cdots E_1(w_m^{(1)}) \cdots \\ &\quad \times \cdots F_t(z_1^{(t)}) \cdots F_t(z_{i-1}^{(t)}) E_t(w_1^{(t)}) \cdots E_t(w_{j-1}^{(t)}) \\ &\quad \times H_t(z) E_t(w_{j+1}^{(t)}) \cdots E_t(w_m^{(t)}) F_t(z_{i+1}^{(t)}) \cdots F_t(z_n^{(t)}) \cdots \\ &\quad \times \cdots E_1(w_1^{(N)}) \cdots E_1(w_m^{(N)}) F_1(z_1^{(N)}) \cdots F_1(z_n^{(N)}) \\ &\quad \times \frac{\prod_{q=1}^N \prod_{1 \leq k < l \leq n} \left[u_k^{(q)} - u_l^{(q)} \right]_r \left[u_l^{(q)} - u_k^{(q)} - 1 \right]_r}{\prod_{q=1}^{N-1} \prod_{k,l=1}^n \left[u_k^{(q)} - u_l^{(q+1)} + 1 - \frac{S}{N} \right]_r \prod_{k,l=1}^n \left[u_k^{(1)} - u_l^{(N)} + \frac{S}{N} \right]_r} \end{aligned}$$

$$\begin{aligned}
& \times \frac{\prod_{q=1}^N \prod_{1 \leq k < l \leq m} \left[v_k^{(q)} - v_l^{(q)} \right]_{-r^*} \left[v_l^{(q)} - v_k^{(q)} - 1 \right]_{-r^*}}{\prod_{q=1}^{N-1} \prod_{k,l=1}^n \left[v_k^{(q)} - v_l^{(q+1)} + 1 - \frac{s}{N} \right]_{-r^*} \prod_{k,l=1}^n \left[v_k^{(1)} - v_l^{(N)} + \frac{s}{N} \right]_{-r^*}} \\
& \times \vartheta \left(\sum_{k=1}^n u_k^{(1)} \middle| \sum_{k=1}^n u_k^{(2)} \middle| \cdots \middle| \sum_{k=1}^n u_k^{(N)} \right) \vartheta \left(\sum_{k=1}^m v_k^{(1)} \middle| \sum_{k=1}^m v_k^{(2)} \middle| \cdots \middle| \sum_{k=1}^m v_k^{(N)} \right) \bigg|_{v_j = u - \frac{r^*}{2}}.
\end{aligned} \tag{4.77}$$

Here the contours $C_{i,j}^{(t)}$ and $\tilde{C}_{i,j}^{(t)}$ are characterized by

$$\begin{aligned}
C_{i,j}^{(t)} : & \quad |x^{2-\frac{2s}{N}} z_k^{(t-1)}|, |x^{\frac{2s}{N}} z_k^{(t+1)}| < |z_i^{(t)}| < |x^{-2r-\frac{2s}{N}} z_k^{(t-1)}|, |x^{-2-2r+\frac{2s}{N}} z_k^{(t+1)}|, (1 \leq k \leq n) \\
& \quad |x^{-2r+3-\frac{2s}{N}} w_l^{(t-1)}|, |x^{-2r+1+\frac{2s}{N}} w_l^{(t+1)}| < |z_i^{(t)}| < |x^{-1-\frac{2s}{N}} w_l^{(t-1)}|, |x^{-3+\frac{2s}{N}} w_l^{(t+1)}|, (1 \leq l \leq m), \\
& \quad |x^{\frac{2s}{N}} z_l^{(q+1)}| < |z_k^{(q)}| < |x^{-2+\frac{2s}{N}} z_l^{(q+1)}|, (1 \leq q \leq N; (q, k) \neq (t, i), (q, l) \neq (t-1, i)), \\
& \quad |x^{\frac{2s}{N}} w_l^{(q+1)}| < |w_k^{(q)}| < |x^{-2+\frac{2s}{N}} w_l^{(q+1)}|, (1 \leq q \leq N; (q, k) \neq (t, j), (q, l) \neq (t-1, j)),
\end{aligned} \tag{4.78}$$

$$\begin{aligned}
\tilde{C}^{(t)} : & \quad |x^{2r+2-\frac{2s}{N}} z_k^{(t-1)}|, |x^{2r+\frac{2s}{N}} z_k^{(t+1)}| < |z_i^{(t)}| < |x^{-\frac{2s}{N}} z_k^{(t-1)}|, |x^{-2+\frac{2s}{N}} z_k^{(t+1)}|, (1 \leq k \leq n), \\
& \quad |x^{3-\frac{2s}{N}} w_l^{(t-1)}|, |x^{1+\frac{2s}{N}} w_l^{(t+1)}| < |z_i^{(t)}| < |x^{2r-1-\frac{2s}{N}} w_l^{(t-1)}|, |x^{2r-3+\frac{2s}{N}} w_l^{(t+1)}|, (1 \leq l \leq m), \\
& \quad |x^{\frac{2s}{N}} z_l^{(q+1)}| < |z_k^{(q)}| < |x^{-2+\frac{2s}{N}} z_l^{(q+1)}|, (1 \leq q \leq N; (q, k) \neq (t, i), (q, l) \neq (t-1, i)), \\
& \quad |x^{\frac{2s}{N}} w_l^{(q+1)}| < |w_k^{(q)}| < |x^{-2+\frac{2s}{N}} w_l^{(q+1)}|, (1 \leq q \leq N; (q, k) \neq (t, j), (q, l) \neq (t-1, j)).
\end{aligned} \tag{4.79}$$

Let us change the variable $z_i^{(t)} \rightarrow x^{2r} z_i^{(t)}$ of the integrand $B^{(t)}(x^{-r} z_i^{(t)}; \{z_k^{(q)}\}_{\substack{1 \leq q \leq N \\ 1 \leq k \leq n}}, \{w_k^{(q)}\}_{\substack{1 \leq q \leq N \\ 1 \leq k \neq j \leq m}})$ and the contour $C_{i,j}^{(t)}$. Using periodic condition of theta function $[u+r]_r = -[u]_r$, we have $B^{(t)}(x^r z_i^{(t)}; \{z_k^{(q)}\}_{\substack{1 \leq q \leq N \\ 1 \leq k \leq n}}, \{w_k^{(q)}\}_{\substack{1 \leq q \leq N \\ 1 \leq k \neq j \leq m}})$ and the contour $\tilde{C}_{i,j}^{(t)}$. Hence we have

$$\begin{aligned}
& \prod_{q=1}^N \prod_{k=1}^n \prod_{p=1}^N \prod_{\substack{l=1 \\ l \neq t}}^m \oint_{C_{i,j}^{(t)}} \frac{dz^{(q)}_k}{2\pi\sqrt{-1}z_k^{(q)}} \frac{dw^{(p)}_l}{2\pi\sqrt{-1}w_l^{(p)}} B_{i,j}^{(t)} \left(x^{-r} z_i^{(t)}; \{z_j^{(q)}\}_{\substack{1 \leq q \leq N \\ 1 \leq k \leq n}}, \{w^{(q)}\}_{\substack{1 \leq q \leq N \\ 1 \leq l \neq j \leq m}} \right) \\
& = \prod_{q=1}^N \prod_{k=1}^n \prod_{p=1}^N \prod_{\substack{l=1 \\ l \neq t}}^m \oint_{\tilde{C}_{i,j}^{(t)}} \frac{dz^{(q)}_k}{2\pi\sqrt{-1}z_k^{(q)}} \frac{dw^{(p)}_l}{2\pi\sqrt{-1}w_l^{(p)}} B_{i,j}^{(t)} \left(x^r z_i^{(t)}; \{z_j^{(q)}\}_{\substack{1 \leq q \leq N \\ 1 \leq k \leq n}}, \{w^{(q)}\}_{\substack{1 \leq q \leq N \\ 1 \leq l \neq j \leq m}} \right).
\end{aligned} \tag{4.80}$$

Therefore we have shown the commutation relation $[\mathcal{G}_m^*, \mathcal{G}_n] = 0$. Generalization to

generic parameter $0 < \text{Re}(r) < 1$ and $s \in \mathbb{C}$ should be understood as analytic continuation. Q.E.D.

5 Dynkin-Automorphism Invariance

In this section we consider the Dynkin-automorphism invariance of the integrals of motion.

5.1 Dynkin-Automorphism Invariance

The integrals of motion are invariant under the action of the Dynkin-automorphism.

Theorem 5.1 *The local integrals of motion $\mathcal{I}_n, \mathcal{I}_n^*$ are invariant under the action of Dynkin-automorphism η*

$$\eta(\mathcal{I}_n) = \mathcal{I}_n, \quad \eta(\mathcal{I}_n^*) = \mathcal{I}_n^*, \quad (n \in \mathbb{N}). \quad (5.1)$$

Theorem 5.2 *The nonlocal integrals of motion $\mathcal{G}_n, \mathcal{G}_n^*$ are invariant under the action of Dynkin-automorphism η*

$$\eta(\mathcal{G}_n) = \mathcal{G}_n, \quad \eta(\mathcal{G}_n^*) = \mathcal{G}_n^*, \quad (n \in \mathbb{N}). \quad (5.2)$$

This theorem plays an important role in proof of the commutation relation $[\mathcal{I}_m, \mathcal{G}_n] = 0$.

5.2 Proof of Dynkin-Automorphism Invariance $\eta(\mathcal{I}_n) = \mathcal{I}_n$

In this section we show Dynkin-Automorphism Invariance $\eta(\mathcal{I}_n) = \mathcal{I}_n$, by using Laurent series formulae $\mathcal{I}_n = [\prod_{j < k} s(z_k/z_j) \mathcal{O}_n(z_1, \dots, z_n)]$. We have $\eta^N = id$. Let us set the functions $h_{J,K}^{\eta^p, \eta^q}(z)$ for $0 \leq J \leq K \leq N$, $0 \leq p \leq J-1$, $0 \leq q \leq K-1$,

$$\begin{aligned} h_{J,K}^{\eta^p, \eta^q}(z) &= \prod_{j=1}^{J-p} \prod_{k=1}^{K-q} h_{11}(x^{-K+J+2(k-j)+\frac{2s}{N}(q-p)} z) \prod_{j=J-p+1}^J \prod_{k=K-q+1}^K h_{11}(x^{-K+J+2(k-j)+\frac{2s}{N}(q-p)} z) \\ &\times \prod_{j=1}^{J-p} \prod_{k=K-q+1}^K h_{11}(x^{-K+J+2(k-j)+\frac{2s}{N}(q-p)-2s} z) \prod_{j=J-p+1}^J \prod_{k=1}^{K-q} h_{11}(x^{-K+J+2(k-j)+\frac{2s}{N}(q-p)+2s} z). \end{aligned} \quad (5.3)$$

Here we have set $h_{1,1}(z) = h(u)$ for $z = x^{2u}$. We use notation $h_{J,K}^{\eta^0, \eta^0}(z) = h_{J,K}^{id, id}(z) = h_{J,K}(z)$.

Proof of Theorem 5.1 Let us study from the invariance of $\mathcal{I}_2 = [s(z_2/z_1)\mathcal{O}_2(z_1, z_2)]_{1, z_1 z_2}$. The action of η is given by

$$\begin{aligned} & \eta([h_{1,1}(z_2/z_1)T_1(z_1)T_1(z_2)]_{1, z_1, z_2}) \\ &= [h_{1,1}(z_2/z_1)T_1(z_1)T_1(z_2)]_{1, z_1, z_2} + [h_{1,1}(z_2/z_1)(\Lambda_1(x^{-s}z_1) - \Lambda_1(x^s z_1)) \sum_{j=2}^N \Lambda_j(x^s z_2)]_{1, z_1, z_2} \\ &+ [h_{1,1}(z_2/z_1) \sum_{j=2}^N \Lambda_j(x^s z_1)(\Lambda_1(x^{-s}z_2) - \Lambda_1(x^s z_2))]_{1, z_1, z_2}. \end{aligned} \quad (5.4)$$

By using the relation

$$\begin{aligned} & h_{1,1}(z) - h_{1,1}(x^{2s}z) = c_{11}(\delta(x^2z) - \delta(x^{2r-2+2s}z)), \\ & c_{11} = -\frac{(x^2; x^{2s})_\infty (x^{2r-2}; x^{2s})_\infty (x^{2s-2}; x^{2s})_\infty (x^{2s-2r+2}; x^{2s})_\infty}{(x^{2r-4}; x^{2s})_\infty (x^{2s}; x^{2s})_\infty (x^{2s}; x^{2s})_\infty (x^{2s-2r+4}; x^{2s})_\infty}, \end{aligned} \quad (5.5)$$

we have

$$\begin{aligned} & [h_{1,1}(z_2/z_1)\Lambda_1(x^{-s}z_1) \sum_{j=2}^N \Lambda_j(x^s z_2)]_{1, z_1 z_2} = [h_{1,1}(z_2/z_1)\Lambda_1(z_1) \sum_{j=2}^N \Lambda_j(z_2)]_{1, z_1 z_2} \\ & + cs_{11}(x^{-2})[\delta(x^2 z_2/z_1) : \Lambda_1(x^{-2s}z_1) \sum_{j=2}^N \Lambda_j(x^{-2}z_1) :]_{1, z_1 z_2}, \end{aligned} \quad (5.6)$$

$$\begin{aligned} & [h_{1,1}(z_2/z_1) \sum_{j=2}^N \Lambda_j(x^s z_1)\Lambda_1(x^{-s}z_2)]_{1, z_1 z_2} = [h_{1,1}(z_2/z_1) \sum_{j=2}^N \Lambda_j(z_1)\Lambda_1(z_2)]_{1, z_1 z_2} \\ & - cs_{11}(x^{-2})[\delta(x^{2-2s}z_2/z_1) : \Lambda_1(x^{-2}z_1) \sum_{j=2}^N \Lambda_j(z_1) :]_{1, z_1 z_2}, \end{aligned} \quad (5.7)$$

Here we have used

$$\delta(x^{2r-2+2s}z_2/z_1)\Lambda_1(x^{-s}z_1)\Lambda_j(x^s z_2) = \delta(x^{2r-2}z_2/z_1)\Lambda_j(x^s z_1)\Lambda_1(x^{-s}z_2) = 0. \quad (5.8)$$

Summing up every terms, we have

$$\begin{aligned} & \eta([h_{1,1}(z_2/z_1)T_1(z_1)T_1(z_2)]_{1, z_1, z_2}) = [h_{1,1}(z_2/z_1)T_1(z_1)T_1(z_2)]_{1, z_1, z_2} \\ & + cs(x^{-2})[\delta(x^2 z_2/z_1)\eta(T_2(x^{-1}z_1))]_{1, z_1 z_2} - cs(x^{-2})[\delta(x^2 z_2/z_1)T_2(x^{-1}z_1)]_{1, z_1 z_2}. \end{aligned} \quad (5.9)$$

Summing up every terms of $\eta([s(z_2/z_1)\mathcal{O}_2(z_1, z_2)]_{1, z_1 z_2})$, we conclude $\eta(\mathcal{I}_2) = \mathcal{I}_2$. Next we study $\eta(\mathcal{I}_3) = \mathcal{I}_3$. We use weakly sense equation for the basic operator $\Lambda_j(z)$

$$\begin{aligned} & g_{1,1} \left(\frac{z_2}{z_1} \right) \Lambda_j(z_1) \Lambda_i(z_2) - g_{1,1} \left(\frac{z_1}{z_2} \right) \Lambda_i(z_2) \Lambda_j(z_1) \\ & \sim c\delta \left(\frac{x^2 z_2}{z_1} \right) : \Lambda_i(z_2) \Lambda_j(z_1) :, (i < j). \end{aligned} \quad (5.10)$$

We have

$$\begin{aligned} & \eta \left(\left[\prod_{1 \leq j < k \leq 3} h_{1,1}(z_k/z_j) T_1(z_1) T_1(z_2) T_1(z_3) \right]_{1, z_1 z_2 z_3} \right) \\ & = \left[\prod_{1 \leq j < k \leq 3} h_{1,1}(z_k/z_j) T_1(z_1) T_1(z_2) T_1(z_3) \right. \\ & \quad - cs(x^{-2}) h_{1,2}(x^{-1} z_2/z_1) T_1(z_1) T_2(x^{-1} z_2) \delta(x^2 z_3/z_2) \\ & \quad - cs(x^{-2}) h_{1,2}(x^{-1} z_1/z_2) T_1(z_2) T_2(x^{-1} z_1) \delta(x^2 z_3/z_1) \\ & \quad - cs(x^{-2}) h_{1,2}(x^{-1} z_1/z_3) T_1(z_3) T_2(x^{-1} z_1) \delta(x^2 z_2/z_1) \\ & \quad + cs(x^{-2}) h_{1,2}^{id,\eta}(x^{-1} z_2/z_1) T_1(z_1) \eta(T_2(x^{-1} z_2)) \delta(x^2 z_3/z_2) \\ & \quad + cs(x^{-2}) h_{1,2}^{id,\eta}(x^{-1} z_1/z_2) T_1(z_2) \eta(T_2(x^{-1} z_1)) \delta(x^2 z_3/z_1) \\ & \quad + cs(x^{-2}) h_{1,2}^{id,\eta}(x^{-1} z_1/z_3) T_1(z_3) \eta(T_2(x^{-1} z_1)) \delta(x^2 z_2/z_1) \\ & \quad + c^2 s(x^{-2})^2 s(x^{-4}) \Delta(x^3) \delta(x^2 z_1/z_2) \delta(x^2 z_3/z_1) \eta^2(T_3(z_1)) \\ & \quad + c^2 s(x^{-2})^2 s(x^{-4}) \Delta(x^3) \delta(x^2 z_1/z_3) \delta(x^2 z_2/z_1) \eta^2(T_3(z_1)) \\ & \quad + c^2 s(x^{-2})^2 s(x^{-4}) \Delta(x^3) \delta(x^2 z_2/z_1) \delta(x^2 z_3/z_2) \eta^2(T_3(z_2)) \\ & \quad + c^2 s(x^{-2})^2 s(x^{-4}) \Delta(x^3) (\delta(x^2 z_1/z_2) \delta(x^2 z_3/z_1) + \delta(x^2 z_1/z_3) \delta(x^2 z_2/z_1)) T_3(z_1) \\ & \quad \left. - c^2 s(x^{-2})^2 s(x^{-4}) \Delta(x^3) (\delta(x^2 z_1/z_2) \delta(x^2 z_3/z_1) + \delta(x^2 z_1/z_3) \delta(x^2 z_2/z_1)) \eta(T_3(z_1)) \right]_{1, z_1 z_2 z_3}. \end{aligned} \quad (5.11)$$

$$\begin{aligned} & \eta([cs(x^{-2}) h_{1,2}(x^{-1} z_2/z_1) T_1(z_1) T_2(x^{-1} z_2) \delta(x^2 z_3/z_2)]_{1, z_1 z_2 z_3}) \\ & = [cs(x^{-2}) h_{1,2}^{id,\eta}(x^{-1} z_2/z_1) T_1(z_1) \eta(T_2(x^{-1} z_2)) \delta(x^2 z_3/z_2) \\ & \quad + c^2 s(x^{-2})^2 s(x^{-4}) \Delta(x^3) \delta(x^2 z_3/z_2) \delta(x^2 z_2/z_1) \eta^2(T_3(z_2))]_{1, z_1 z_2 z_3}, \end{aligned} \quad (5.12)$$

$$\begin{aligned} & \eta([\delta(x^2 z_1/z_2) \delta(x^2 z_3/z_1) T_3(z_1)]_{1, z_1 z_2 z_3}) = [\delta(x^2 z_1/z_2) \delta(x^2 z_3/z_1) \eta(T_3(z_1))]_{1, z_1 z_2 z_3}. \end{aligned} \quad (5.13)$$

Summing up every term of $\eta([s(z_2/z_1)s(z_3/z_1)s(z_3/z_2)\mathcal{O}_3(z_1, z_2, z_3)]_{1, z_1 z_2 z_3})$, we have $\eta(\mathcal{I}_3) =$

\mathcal{I}_3 . We consider the case of general \mathcal{I}_n . The action of η is given by

$$\begin{aligned}
& \eta \left(\left[\prod_{1 \leq j < k \leq n} h(z_k/z_j) T_1(z_1) T_1(z_2) T_1(z_3) \cdots T_1(z_n) \right]_{1, z_1 \cdots z_n} \right) = \sum_{\substack{\alpha_1, \alpha_2, \dots, \alpha_N \geq 0 \\ \alpha_1 + 2\alpha_2 + \cdots + N\alpha_N = n}} \\
& \times \sum_{\substack{\alpha_1^{(1)} = \alpha_1 \\ \alpha_2^{(1)}, \alpha_2^{(2)} \geq 0, \alpha_2^{(1)} + \alpha_2^{(2)} = \alpha_2 \\ \alpha_3^{(1)}, \alpha_3^{(2)}, \alpha_3^{(3)} \geq 0, \alpha_3^{(1)} + \alpha_3^{(2)} + \alpha_3^{(3)} = \alpha_3 \\ \alpha_N^{(1)}, \alpha_N^{(2)}, \dots, \alpha_N^{(N)} \geq 0, \alpha_N^{(1)} + \alpha_N^{(2)} + \cdots + \alpha_N^{(N)} = \alpha_N}} \sum_{\substack{\{A_j^{(t,q)}\}_{1 \leq q \leq t \leq N, 1 \leq j \leq \alpha_t^{(q)}} \\ A_j^{(t,q)} \subset \{1, 2, \dots, n\}, |A_j^{(t,q)}| = t, \cup_{t=1}^N \cup_{q=1}^t \cup_{j=1}^{\alpha_t^{(q)}} A_j^{(t,q)} \\ \text{Min}(A_1^{(t,q)}) < \text{Min}(A_2^{(t,q)}) < \cdots < \text{Min}(A_{\alpha_s^{(t)}}^{(t,q)})}} \\
& \times \sum_{\substack{\widetilde{A_j^{(3,3)}} \subset A_j^{(3,3)}, |\widetilde{A_j^{(3,3)}}| = 2 \\ \widetilde{A_j^{(4,3)}} \subset A_j^{(4,3)}, \widetilde{A_j^{(4,4)}} \subset A_j^{(4,4)}, |\widetilde{A_j^{(4,3)}}| = 3, |\widetilde{A_j^{(4,4)}}| = 2 \\ \dots \\ \widetilde{A_j^{(N,3)}} \subset A_j^{(N,3)}, \widetilde{A_j^{(N,4)}} \subset A_j^{(N,4)}, \dots, \widetilde{A_j^{(N,N)}} \subset A_j^{(N,N)}, |\widetilde{A_j^{(N,3)}}| = N-2, |\widetilde{A_j^{(N,4)}}| = N-3, \dots, |\widetilde{A_j^{(N,N)}}| = 2}} \\
& \times (-1)^{\sum_{t=1}^{\lfloor \frac{N}{2} \rfloor} \sum_{u=1}^t \alpha_{2t}^{(2u-1)} + \sum_{t=1}^{\lfloor \frac{N-1}{2} \rfloor} \sum_{u=1}^t \alpha_{2t+1}^{(2u)}} \prod_{t=2}^N \left(c^{t-1} \prod_{u=1}^{t-2} \Delta(x^{2u+1})^{t-u-1} \prod_{u=1}^{t-1} s(x^{-2u})^{t-u} \right)^{\alpha_t} \\
& \times \left[\prod_{t=1}^N \prod_{q=1}^t \prod_{\substack{j < k \\ j, k \in A_{\text{Min}}^{(t,q)}}} h_{t,t}^{\eta^{q-1}, \eta^{q-1}}(z_k/z_j) \right. \\
& \times \prod_{1 \leq t < u \leq N} \prod_{q=1}^t \prod_{p=1}^u \prod_{j \in A_{\text{Min}}^{(t,q)}} \prod_{k \in A_{\text{Min}}^{(u,p)}} h_{t,u}^{\eta^{q-1}, \eta^{p-1}}(x^{u-t-2[\frac{u}{2}] + 2[\frac{t}{2}]} z_k/z_j) \\
& \times \left\{ \prod_{j \in \overrightarrow{A_{\text{Min}}^{(1,1)}}} T_1(z_j) \right\} \left\{ \prod_{j \in \overrightarrow{A_{\text{Min}}^{(2,1)}}} T_2(x^{-1} z_j) \prod_{j \in \overrightarrow{A_{\text{Min}}^{(2,2)}}} \eta(T_2(x^{-1} z_j)) \right\} \cdots \\
& \times \left\{ \prod_{j \in \overrightarrow{A_{\text{Min}}^{(N,1)}}} T_N(x^{-1+N-2[\frac{N}{2}]} z_j) \cdots \prod_{j \in \overrightarrow{A_{\text{Min}}^{(N,N)}}} \eta^{N-1}(T_N(x^{-1+N-2[\frac{N}{2}]} z_j)) \right\} \\
& \times \prod_{q=1}^2 \prod_{j=1}^{\alpha_2^{(q)}} \delta \left(\frac{x^2 z_{j_2}}{z_{j_1}} \right) \prod_{t=3}^N \prod_{q=1}^2 \prod_{\substack{j=1 \\ j_1 = A_{j,1}^{(t,q)} \\ \dots \\ j_t = A_{j,t}^{(t,q)}}}^{\alpha_t^{(q)}} \sum_{\substack{\sigma \in S_t \\ \sigma(1)=1}} \prod_{\substack{u=1 \\ u \neq [\frac{t}{2}]+1}}^t \delta \left(\frac{x^2 z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}} \right) \\
& \times \prod_{t=3}^N \prod_{q=3}^t \prod_{\substack{j=1 \\ j_1 = A_{j,1}^{(t,q)} \\ \dots \\ j_{t-q+2} = A_{j,t-q+2}^{(t,q)}}}^{\alpha_t^{(q)}} \prod_{\substack{k_1, k_2, \dots, k_{q-2} \in A_j^{(t,q)} - A_j^{(t,q)} \\ k_1 < k_2 < \cdots < k_{q-2}}} \sum_{\substack{\sigma \in S_{t-q+2} \\ \sigma(1)=1}} \prod_{\substack{u=1 \\ u \neq [\frac{t-q}{2}]+2}}^{t-q+2} \delta \left(\frac{x^2 z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}} \right) \sum_{\tau \in S_{q-2}}
\end{aligned}$$

$$\begin{aligned}
& \times \prod_{t-q \in 2\mathbb{Z}} \delta \left(\frac{x^2 z_{k_{\tau(\lfloor \frac{q+1}{2} \rfloor)}}}{z_{j_{\sigma(\lfloor \frac{t-q}{2} \rfloor + 2)}}} \right) \delta \left(\frac{x^2 z_{j_{\sigma(\lfloor \frac{t-q}{2} \rfloor + 3)}}}{z_{k_{\tau(1)}}} \right) \prod_{u=\lfloor \frac{q+3}{2} \rfloor}^{q-2} \delta \left(\frac{x^2 z_{k_{\tau(u)}}}{z_{k_{\tau(u-1)}}} \right) \prod_{u=2}^{\lfloor \frac{q-1}{2} \rfloor} \delta \left(\frac{x^2 z_{k_{\tau(u-1)}}}{z_{k_{\tau(u)}}} \right) \\
& \times \prod_{t-q+1 \in 2\mathbb{Z}} \delta \left(\frac{x^2 z_{k_{\tau(1)}}}{z_{j_{\sigma(\lfloor \frac{t-q}{2} \rfloor + 2)}}} \right) \delta \left(\frac{x^2 z_{j_{\sigma(\lfloor \frac{t-q}{2} \rfloor + 3)}}}{z_{k_{\tau(\lfloor \frac{q+1}{2} \rfloor)}}} \right) \prod_{u=2}^{\lfloor \frac{q-1}{2} \rfloor} \delta \left(\frac{x^2 z_{k_{\tau(u)}}}{z_{k_{\tau(u-1)}}} \right) \prod_{u=\lfloor \frac{q+3}{2} \rfloor}^{q-2} \delta \left(\frac{x^2 z_{k_{\tau(u-1)}}}{z_{k_{\tau(u)}}} \right).
\end{aligned} \tag{5.14}$$

Here we have set $A_{Min}^{(t,q)} = \{Min(A_1^{(t,q)}), Min(A_2^{(t,q)}), \dots, Min(A_{\alpha_t}^{(t,q)})\}$ for $q = 0, 1$, and have set $\widetilde{A_{Min}^{(t,q)}} = \{Min(\widetilde{A_1^{(t,q)}}), Min(\widetilde{A_2^{(t,q)}}), \dots, Min(\widetilde{A_{\alpha_t}^{(t,q)}})\}$ for $q = 2, \dots, t$. Here we have set $A_{j,1}^{(u,t)}, A_{j,2}^{(u,t)}, \dots, A_{j,u}^{(u,t)}$ such that $A_j^{(u,t)} = \{A_{j,1}^{(u,t)} < A_{j,2}^{(u,t)} < \dots < A_{j,u}^{(u,t)}\}$, and have set $\widetilde{A_{j,1}^{(u,t)}}, \widetilde{A_{j,2}^{(u,t)}}, \dots, \widetilde{A_{j,N+2-t}^{(u,t)}}$ such that $A_j^{(u,t)} = \{\widetilde{A_{j,1}^{(u,t)}} < \widetilde{A_{j,2}^{(u,t)}} < \dots < \widetilde{A_{j,N+2-t}^{(u,t)}}\}$.

We give the action of η for more general case. We prepare notations. Let us set $\beta_1, \beta_2, \dots, \beta_N \geq 0$ such that $\beta_1 + 2\beta_2 + 3\beta_3 + \dots + N\beta_N = n$. Let us set subset $B_j^{(t)} \subset \{1, 2, \dots, n\}$, ($1 \leq t \leq N, 1 \leq j \leq \beta_t$) such that $|B_j^{(t)}| = t$, $\cup_{t=1}^N \cup_{j=1}^{\alpha_t} B_j^{(t)} = \{1, 2, \dots, n\}$ and $Min(B_1^{(t)}) < Min(B_2^{(t)}) < \dots < Min(B_{\alpha_t}^{(t)})$. Let us set the index $B_{j,k}^{(t)} = j_k$ for $B_j^{(t)} = \{j_1, j_2, \dots, j_t | j_1 < j_2 < \dots < j_t\}$, ($1 \leq t \leq N, 1 \leq j \leq \alpha_t$), and $B_{Min}^{(t)} = \{B_{1,1}^{(t)}, B_{2,1}^{(t)}, \dots, B_{\alpha_t,1}^{(t)}\}$. The action of η is given as following.

$$\begin{aligned}
& \eta \left(\left[\prod_{j \in \overrightarrow{B_{Min}^{(1)}}} T_1(z_j) \prod_{j \in \overrightarrow{B_{Min}^{(2)}}} T_2(x^{-1}z_j) \dots \prod_{j \in \overrightarrow{B_{Min}^{(N)}}} T_N(x^{-1+N-2\lfloor \frac{N}{2} \rfloor} z_j) \right. \right. \\
& \times \prod_{t=1}^N \left((-c)^{t-1} \prod_{u=1}^{t-1} \Delta(x^{2u+1})^{t-u-1} \right)^{\beta_t} \prod_{t=2}^N \prod_{\substack{j=1 \\ j_1=B_{j,1}^{(t)} \\ \dots \\ j_t=B_{j,t}^{(t)}}}^{\beta_t} \sum_{\substack{\sigma \in S_t \\ \sigma(1)=1}} \prod_{\substack{u=1 \\ u \neq \lfloor \frac{t}{2} \rfloor + 1}}^t \delta \left(\frac{x^2 z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}} \right) \\
& \times \prod_{t=1}^N \prod_{\substack{j < k \\ j, k \in B_{Min}^{(t)}}} g_{t,t} \left(\frac{z_k}{z_j} \right) \prod_{1 \leq t < u \leq N} \prod_{\substack{j \in B_{Min}^{(t)} \\ k \in B_{Min}^{(u)}}} g_{t,u} \left(x^{u-t-2\lfloor \frac{u}{2} \rfloor + 2\lfloor \frac{t}{2} \rfloor} \frac{z_k}{z_j} \right) \left. \right]_{1, z_1 \dots z_n} \\
& = \sum_{\substack{\alpha_1, \alpha_2, \dots, \alpha_N \geq 0 \\ \alpha_1 + 2\alpha_2 + \dots + N\alpha_N = n}} \sum_{\substack{\alpha_1^{(1)} = \alpha_1 \\ \alpha_2^{(1)}, \alpha_2^{(2)} \geq 0, \alpha_2^{(1)} + \alpha_2^{(2)} = \alpha_2 \\ \alpha_3^{(1)}, \alpha_3^{(2)}, \alpha_3^{(3)} \geq 0, \alpha_3^{(1)} + \alpha_3^{(2)} + \alpha_3^{(3)} = \alpha_3 \\ \dots \\ \alpha_N^{(1)}, \alpha_N^{(2)}, \dots, \alpha_N^{(N)} \geq 0, \alpha_N^{(1)} + \alpha_N^{(2)} + \dots + \alpha_N^{(N)} = \alpha_N}} \sum_{\substack{\{A_j^{(t,q)}\} \\ 1 \leq q \leq t \leq N, 1 \leq j \leq \alpha_t^{(q)} \\ A_j^{(t,q)} \subset \{1, 2, \dots, n\}, |A_j^{(t,q)}| = t, \cup_{t=1}^N \cup_{q=1}^t \cup_{j=1}^{\alpha_t^{(q)}} A_j^{(t,q)} \\ Min(A_1^{(t,q)}) < Min(A_2^{(t,q)}) < \dots < Min(A_{\alpha_t}^{(t,q)})}}
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{\substack{\widetilde{A_j^{(3,3)}} \subset A_j^{(3,3)}, |\widetilde{A_j^{(3,3)}}|=2 \\ \widetilde{A_j^{(4,3)}} \subset A_j^{(4,3)}, \widetilde{A_j^{(4,4)}} \subset A_j^{(4,4)}, |\widetilde{A_j^{(4,3)}}|=3, |\widetilde{A_j^{(4,4)}}|=2 \\ \widetilde{A_j^{(N,3)}} \subset A_j^{(N,3)}, \widetilde{A_j^{(N,4)}} \subset A_j^{(N,4)}, \dots, \widetilde{A_j^{(N,N)}} \subset A_j^{(N,N)}, |\widetilde{A_j^{(N,3)}}|=N-2, |\widetilde{A_j^{(N,4)}}|=N-3, \dots, |\widetilde{A_j^{(N,N)}}|=2}} \\
& \times \sum_{\substack{\{B_j^{(N-1)}\}_{1 \leq j \leq \beta_{N-1}} \subset \{A_j^{(N,2)}\}_{1 \leq j \leq \alpha_N^{(2)}} \quad \{B_j^{(2)}\}_{1 \leq j \leq \beta_2} \subset \{\widetilde{A_j^{(t,t)}}\}_{3 \leq t \leq N, 1 \leq j \leq \alpha_t^{(t)}} \\ \{B_j^{(N)}\}_{1 \leq j \leq \beta_N} \subset \{A_j^{(N,1)}\}_{1 \leq j \leq \alpha_N^{(1)}} \quad \{B_j^{(3)}\}_{1 \leq j \leq \beta_3} \subset \{\widetilde{A_j^{(t,t-1)}}\}_{4 \leq t \leq N, 1 \leq j \leq \alpha_t^{(t-1)}} \\ \dots \\ \{B_j^{(N-2)}\}_{1 \leq j \leq \beta_{N-2}} \subset \{\widetilde{A_j^{(N,3)}}\}_{1 \leq j \leq \alpha_N^{(3)}}}} \sum_{\substack{\{B_j^{(2)}\}_{1 \leq j \leq \beta_2} \subset \{\widetilde{A_j^{(t,t)}}\}_{3 \leq t \leq N, 1 \leq j \leq \alpha_t^{(t)}} \\ \{B_j^{(3)}\}_{1 \leq j \leq \beta_3} \subset \{\widetilde{A_j^{(t,t-1)}}\}_{4 \leq t \leq N, 1 \leq j \leq \alpha_t^{(t-1)}} \\ \dots \\ \{B_j^{(N-2)}\}_{1 \leq j \leq \beta_{N-2}} \subset \{\widetilde{A_j^{(N,3)}}\}_{1 \leq j \leq \alpha_N^{(3)}}}} \\
& \times (-1)^{\sum_{t=1}^{\lfloor \frac{N}{2} \rfloor} \sum_{u=1}^t \alpha_{2t}^{(2u-1)} + \sum_{t=1}^{\lfloor \frac{N-1}{2} \rfloor} \sum_{u=1}^t \alpha_{2t+1}^{(2u)} + \sum_{j=2}^N \beta_j} \prod_{t=2}^N \left(c^{t-1} \prod_{u=1}^{t-2} \Delta(x^{2u+1})^{t-u-1} \prod_{u=1}^{t-1} s(x^{-2u})^{t-u} \right)^{\alpha_t} \\
& \times \left[\prod_{t=1}^N \prod_{q=1}^t \prod_{\substack{j < k \\ j, k \in A_{Min}^{(t,q)}}} h_{t,t}^{\eta^{q-1}, \eta^{q-1}}(z_k/z_j) \right. \\
& \times \prod_{1 \leq t < u \leq N} \prod_{q=1}^t \prod_{p=1}^u \prod_{j \in A_{Min}^{(t,q)}} \prod_{k \in A_{Min}^{(u,p)}} h_{t,u}^{\eta^{q-1}, \eta^{p-1}}(x^{u-t-2\lfloor \frac{u}{2} \rfloor + 2\lfloor \frac{t}{2} \rfloor} z_k/z_j) \\
& \times \left\{ \prod_{j \in \overrightarrow{A_{Min}^{(1,1)}}} T_1(z_j) \right\} \left\{ \prod_{j \in \overrightarrow{A_{Min}^{(2,1)}}} T_2(x^{-1}z_j) \prod_{j \in \overrightarrow{A_{Min}^{(2,2)}}} \eta(T_2(x^{-1}z_j)) \right\} \dots \\
& \times \left\{ \prod_{j \in \overrightarrow{A_{Min}^{(N,1)}}} T_N(x^{-1+N-2\lfloor \frac{N}{2} \rfloor} z_j) \dots \prod_{j \in \overrightarrow{A_{Min}^{(N,N)}}} \eta^{N-1}(T_N(x^{-1+N-2\lfloor \frac{N}{2} \rfloor} z_j)) \right\} \\
& \times \prod_{q=1}^2 \prod_{j=1}^{\alpha_2^{(q)}} \delta\left(\frac{x^2 z_{j_2}}{z_{j_1}}\right) \prod_{t=3}^N \prod_{q=1}^2 \prod_{\substack{j=1 \\ j_1 = A_{j,1}^{(t,q)} \\ \dots \\ j_t = A_{j,t}^{(t,q)}}}^{\alpha_t^{(q)}} \sum_{\substack{\sigma \in S_t \\ \sigma(1)=1}} \prod_{\substack{u=1 \\ u \neq \lfloor \frac{t}{2} \rfloor + 1}}^t \delta\left(\frac{x^2 z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}}\right) \\
& \times \prod_{t=3}^N \prod_{q=3}^t \prod_{\substack{j=1 \\ j_1 = A_{j,1}^{(t,q)} \\ \dots \\ j_{t-q+2} = A_{j,t-q+2}^{(t,q)}}}^{\alpha_t^{(q)}} \prod_{\substack{k_1, k_2, \dots, k_{q-2} \in A_j^{(t,q)} - \widetilde{A_j^{(t,q)}} \\ k_1 < k_2 < \dots < k_{q-2}}} \sum_{\substack{\sigma \in S_{t-q+2} \\ \sigma(1)=1}} \prod_{\substack{u=1 \\ u \neq \lfloor \frac{t-q}{2} \rfloor + 2}}^{t-q+2} \delta\left(\frac{x^2 z_{j_{\sigma(u+1)}}}{z_{j_{\sigma(u)}}}\right) \sum_{\tau \in S_{q-2}} \\
& \times \prod_{t-q \in 2\mathbb{Z}} \delta\left(\frac{x^2 z_{k_{\tau(\lfloor \frac{q+1}{2} \rfloor)}}}{z_{j_{\sigma(\lfloor \frac{t-q}{2} \rfloor + 2)}}}\right) \delta\left(\frac{x^2 z_{j_{\sigma(\lfloor \frac{t-q}{2} \rfloor + 3)}}}{z_{k_{\tau(1)}}}\right) \prod_{u=\lfloor \frac{q+3}{2} \rfloor}^{q-2} \delta\left(\frac{x^2 z_{k_{\tau(u)}}}{z_{k_{\tau(u-1)}}}\right) \prod_{u=2}^{\lfloor \frac{q-1}{2} \rfloor} \delta\left(\frac{x^2 z_{k_{\tau(u-1)}}}{z_{k_{\tau(u)}}}\right)
\end{aligned}$$

$$\times \prod_{t-q+1 \in 2\mathbb{Z}} \delta \left(\frac{x^2 z_{k_{\tau(1)}}}{z_{\sigma([\frac{t-q}{2}]+2)}^j} \right) \delta \left(\frac{x^2 z_j^{\sigma([\frac{t-q}{2}]+3)}}{z_{\tau([\frac{q+1}{2}])}^k} \right) \prod_{u=2}^{[\frac{q-1}{2}] \delta \left(\frac{x^2 z_{k_{\tau(u)}}}{z_{k_{\tau(u-1)}}} \right) \prod_{u=[\frac{q+3}{2}]^{q-2} \delta \left(\frac{x^2 z_{k_{\tau(u-1)}}}{z_{k_{\tau(u)}}} \right). \quad (5.15)$$

We note that differences between the equations (5.14) and (5.15) are the signature and the restriction condition

$$\sum_{\substack{\{B_j^{(N-1)}\}_{1 \leq j \leq \beta_{N-1}} \subset \{A_j^{(N,2)}\}_{1 \leq j \leq \alpha_N^{(2)}} \\ \{B_j^{(N)}\}_{1 \leq j \leq \beta_N} \subset \{A_j^{(N,1)}\}_{1 \leq j \leq \alpha_N^{(1)}}}} \sum_{\substack{\{B_j^{(2)}\}_{1 \leq j \leq \beta_2} \subset \{A_j^{(t,t)}\}_{3 \leq t \leq N, 1 \leq j \leq \alpha_t^{(t)}} \\ \{B_j^{(3)}\}_{1 \leq j \leq \beta_3} \subset \{A_j^{(t,t-1)}\}_{4 \leq t \leq N, 1 \leq j \leq \alpha_t^{(t-1)}} \\ \dots \\ \{B_j^{(N-2)}\}_{1 \leq j \leq \beta_{N-2}} \subset \{A_j^{(N,3)}\}_{1 \leq j \leq \alpha_N^{(3)}}}} .$$

Hence, summing up every terms of $\eta([\prod_{1 \leq j < k \leq n} s(z_k/z_j) \mathcal{O}_n(z_1, z_2, \dots, z_n)]_{1, z_1 \dots z_n})$, we have shown $\eta(\mathcal{I}_n) = \mathcal{I}_n$. Q.E.D.

5.3 Proof of Dynkin-Automorphism Invariance $\eta(\mathcal{G}_n) = \mathcal{G}_n$

In this section we show Dynkin-automorphism invariance $\eta(\mathcal{G}_n) = \mathcal{G}_n$.

Proof of Theorem 5.2 For reader's convenience, we explain $\eta(\mathcal{G}_1) = \mathcal{G}_1$ at first. We have the action of η as following.

$$\begin{aligned} & \eta \left(\prod_{t=1}^N \oint_C \frac{dz^{(t)}}{2\pi\sqrt{-1}z^{(t)}} \frac{F_1(z^{(1)})F_2(z^{(2)}) \dots F_N(z^{(N)}) \vartheta(u^{(1)}|u^{(2)}|\dots|u^{(N)})}{\prod_{t=1}^{N-1} \left[u^{(t)} - u^{(t+1)} + 1 - \frac{s}{N} \right]_r \left[u^{(1)} - u^{(N)} + \frac{s}{N} \right]_r} \right) \\ &= \prod_{t=1}^N \oint_C \frac{dz^{(t)}}{2\pi\sqrt{-1}z^{(t)}} \frac{\eta(\vartheta(u^{(1)}|u^{(2)}|\dots|u^{(N)}))}{\prod_{t=2}^{N-1} \left[u^{(t)} - u^{(t+1)} + 1 - \frac{s}{N} \right]_r \left[u^{(N)} - u^{(1)} + 1 - \frac{s}{N} \right]_r \left[u^{(2)} - u^{(1)} + \frac{s}{N} \right]_r} \\ &\times F_2(z^{(1)})F_3(z^{(2)}) \dots F_N(z^{(N-1)})F_1(z^{(N)}). \end{aligned} \quad (5.16)$$

Here we have used

$$\eta(F_1(z_1)F_2(z_2) \dots F_N(z_N)) = F_2(z_1) \dots F_N(z_{N-1})F_1(z_N). \quad (5.17)$$

Let us change variables $u^{(1)} \rightarrow u^{(N)}, u^{(2)} \rightarrow u^{(1)}, u^{(2)} \rightarrow u^{(1)}, \dots, u^{(N)} \rightarrow u^{(N-1)}$, and move $F_1(z^{(1)})$ from the right to the left. We have

$$\prod_{t=1}^N \oint_C \frac{dz^{(t)}}{2\pi\sqrt{-1}z^{(t)}} \frac{F_1(z^{(1)})F_2(z^{(2)}) \cdots F_N(z^{(N)})\eta(\vartheta(u^{(2)}|u^{(3)}|\cdots|u^{(N)}|u^{(1)}))}{\prod_{t=1}^{N-1} \left[u^{(t)} - u^{(t+1)} + 1 - \frac{S}{N}\right]_r \left[u^{(1)} - u^{(N)} + \frac{S}{N}\right]_r}. \quad (5.18)$$

We conclude $\eta(\mathcal{G}_1) = \mathcal{G}_1$ from theta property $\eta(\vartheta(u^{(2)}|u^{(3)}|\cdots|u^{(N)}|u^{(1)})) = \vartheta(u^{(1)}|u^{(2)}|\cdots|u^{(N)})$.

Let us show $\eta(\mathcal{G}_m) = \mathcal{G}_m$. After changing the variables $u_j^{(1)} \rightarrow u_j^{(N)}, u_j^{(2)} \rightarrow u_j^{(1)}, u_j^{(3)} \rightarrow u_j^{(2)}, \dots, u_j^{(N)} \rightarrow u_j^{(N-1)}$, we have $\eta(\mathcal{G}_m)$ as following.

$$\begin{aligned} & \prod_{t=1}^N \prod_{j=1}^m \oint_C \frac{dz_j^{(t)}}{2\pi\sqrt{-1}z_j^{(t)}} F_2(z_1^{(1)}) \cdots F_2(z_m^{(1)}) F_3(z_1^{(2)}) \cdots F_3(z_m^{(2)}) \cdots F_N(z_1^{(N-1)}) \cdots F_N(z_m^{(N-1)}) \\ & \times F_1(z_1^{(N)}) \cdots F_1(z_m^{(N)}) \eta \left(\vartheta \left(\sum_{j=1}^m u_j^{(1)} \middle| \sum_{j=1}^m u_j^{(2)} \middle| \cdots \middle| \sum_{j=1}^m u_j^{(N)} \right) \right) \\ & \times \frac{\prod_{t=1}^N \prod_{1 \leq i < j \leq m} [u_i^{(t)} - u_j^{(t)}]_r [u_j^{(t)} - u_i^{(t)} - 1]_r}{\prod_{t=2}^{N-1} \prod_{i,j=1}^m \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{S}{N}\right]_r \prod_{i,j=1}^m \left[u_i^{(N)} - u_j^{(1)} + 1 - \frac{S}{N}\right]_r \prod_{i,j=1}^m \left[u_i^{(1)} - u_j^{(2)} + \frac{S}{N}\right]_r}. \end{aligned} \quad (5.19)$$

Here we have used

$$\begin{aligned} & \eta(F_1(z_1^{(1)}) \cdots F_1(z_m^{(1)}) \cdots F_{N-1}(z_1^{(N-1)}) \cdots F_{N-1}(z_m^{(N-1)}) F_N(z_1^{(N)}) \cdots F_N(z_m^{(N)})) \\ & = F_2(z_1^{(1)}) \cdots F_2(z_m^{(1)}) F_3(z_1^{(2)}) \cdots F_3(z_m^{(2)}) \cdots F_N(z_1^{(N-1)}) \cdots F_N(z_m^{(N-1)}) F_1(z_1^{(N)}) \cdots F_1(z_m^{(N)}). \end{aligned} \quad (5.20)$$

Let us change the variables $u_j^{(1)} \rightarrow u_j^{(2)}, u_j^{(2)} \rightarrow u_j^{(3)}, \dots, u_j^{(N-1)} \rightarrow u_j^{(N)}, u_j^{(N)} \rightarrow u_j^{(1)}$, and move $F_1(z_1^{(N)}) \cdots F_1(z_m^{(N)})$ from the right to the left. We have

$$\begin{aligned} & \prod_{t=1}^N \prod_{j=1}^m \oint_C \frac{dz_j^{(t)}}{2\pi\sqrt{-1}z_j^{(t)}} F_1(z_1^{(1)}) \cdots F_1(z_m^{(1)}) F_2(z_1^{(2)}) \cdots F_2(z_m^{(2)}) \cdots F_N(z_1^{(N)}) \cdots F_N(z_m^{(N)}) \\ & \times \frac{\prod_{t=1}^N \prod_{1 \leq i < j \leq m} [u_i^{(t)} - u_j^{(t)}]_r [u_j^{(t)} - u_i^{(t)} - 1]_r}{\prod_{t=1}^{N-1} \prod_{i,j=1}^m \left[u_i^{(t)} - u_j^{(t+1)} + 1 - \frac{S}{N}\right]_r \prod_{i,j=1}^m \left[u_i^{(1)} - u_j^{(N)} + \frac{S}{N}\right]_r} \\ & \times \eta \left(\vartheta \left(\sum_{j=1}^m u_j^{(2)} \middle| \sum_{j=1}^m u_j^{(3)} \middle| \cdots \middle| \sum_{j=1}^m u_j^{(N)} \middle| \sum_{j=1}^m u_j^{(1)} \right) \right). \end{aligned} \quad (5.21)$$

We conclude $\eta(\mathcal{G}_m) = \mathcal{G}_m$ from $\eta(\vartheta(u^{(1)}|\cdots|u^{(N)})) = \vartheta(u^{(N)}|u^{(1)}|\cdots|u^{(N-1)})$. Proof of $\eta(\mathcal{G}_m^*) = \mathcal{G}_m^*$ is given in the same manner as above. Q.E.D.

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A Normal Ordering

We summarize the normal orderings of the basic operators. The normal orderings for $\Lambda_j(z)$ are given as followings.

$$\Lambda_i(z_1)\Lambda_i(z_2) = :: (1 - z_2/z_1) \frac{(x^2 z_2/z_1; x^{2s})_\infty (x^{2r+2s-2} z_2/z_1; x^{2s})_\infty (x^{2s-2r} z_2/z_1; x^{2s})_\infty}{(x^{2s-2} z_2/z_1; x^{2s})_\infty (x^{2r} z_2/z_1; x^{2s})_\infty (x^{2-2r} z_2/z_1; x^{2s})_\infty}, \quad (1 \leq i \leq N), \quad (\text{A.1})$$

$$\Lambda_i(z_1)\Lambda_j(z_2) = :: \frac{(x^2 z_2/z_1; x^{2s})_\infty (x^{-2r} z_2/z_1; x^{2s})_\infty (x^{2r-2} z_2/z_1; x^{2s})_\infty}{(x^{-2} z_2/z_1; x^{2s})_\infty (x^{2r} z_2/z_1; x^{2s})_\infty (x^{2-2r} z_2/z_1; x^{2s})_\infty} \quad (1 \leq i < j \leq N), \quad (\text{A.2})$$

$$\Lambda_j(z_1)\Lambda_i(z_2) = :: \frac{(x^{2+2s} z_2/z_1; x^{2s})_\infty (x^{2s-2r} z_2/z_1; x^{2s})_\infty (x^{2s+2r-2} z_2/z_1; x^{2s})_\infty}{(x^{2s-2} z_2/z_1; x^{2s})_\infty (x^{2s+2r} z_2/z_1; x^{2s})_\infty (x^{2s+2-2r} z_2/z_1; x^{2s})_\infty}, \quad (1 \leq i < j \leq N). \quad (\text{A.3})$$

For $N \geq 3$, the normal orderings between $\Lambda_j(z)$ and $E_j(z), F_j(z)$ are given as followings. The normal orderings for $N = 2$ are summarized in appendix in [13].

$$\Lambda_j(z_1)F_j(z_2) = :: x^{-2r^*} \frac{(1 - x^{r-2+\frac{2s}{N}j} z_2/z_1)}{(1 - x^{-r+\frac{2s}{N}j} z_2/z_1)}, \quad (1 \leq j \leq N-1) \quad (\text{A.4})$$

$$F_j(z_1)\Lambda_j(z_2) = :: \frac{(1 - x^{2-r-\frac{2s}{N}j} z_2/z_1)}{(1 - x^{r-\frac{2s}{N}j} z_2/z_1)}, \quad (1 \leq j \leq N-1) \quad (\text{A.5})$$

$$\Lambda_{j+1}(z_1)F_j(z_2) = :: x^{2r^*} \frac{(1 - x^{2-r+\frac{2s}{N}j} z_2/z_1)}{(1 - x^{r+\frac{2s}{N}j} z_2/z_1)}, \quad (1 \leq j \leq N-1) \quad (\text{A.6})$$

$$F_j(z_1)\Lambda_{j+1}(z_2) = :: \frac{(1 - x^{r-2-\frac{2s}{N}j} z_2/z_1)}{(1 - x^{-r-\frac{2s}{N}j} z_2/z_1)}, \quad (1 \leq j \leq N-1) \quad (\text{A.7})$$

$$\Lambda_1(z_1)F_N(z_2) = :: x^{2r^*} \frac{(1 - x^{2-r} z_2/z_1)}{(1 - x^r z_2/z_1)}, \quad (\text{A.8})$$

$$F_N(z_1)\Lambda_1(z_2) = :: \frac{(1 - x^{r-2} z_2/z_1)}{(1 - x^{-r} z_2/z_1)}, \quad (\text{A.9})$$

$$\Lambda_N(z_1)F_N(z_2) = :: x^{-2r^*} \frac{(1 - x^{r-2+2s} z_2/z_1)}{(1 - x^{-r+2s} z_2/z_1)}, \quad (\text{A.10})$$

$$F_N(z_1)\Lambda_N(z_2) = :: \frac{(1 - x^{2-r-2s} z_2/z_1)}{(1 - x^{r-2s} z_2/z_1)}. \quad (\text{A.11})$$

$$\Lambda_j(z_1)E_j(z_2) = :: x^{2r} \frac{(1 - x^{-r-1+\frac{2s}{N}j} z_2/z_1)}{(1 - x^{r-1+\frac{2s}{N}j} z_2/z_1)}, \quad (1 \leq j \leq N-1) \quad (\text{A.12})$$

$$E_j(z_1)\Lambda_j(z_2) = :: \frac{(1 - x^{r+1-\frac{2s}{N}j} z_2/z_1)}{(1 - x^{-r+1-\frac{2s}{N}j} z_2/z_1)}, \quad (1 \leq j \leq N-1) \quad (\text{A.13})$$

$$\Lambda_{j+1}(z_1)E_j(z_2) = :: x^{-2r} \frac{(1 - x^{r+1+\frac{2s}{N}j} z_2/z_1)}{(1 - x^{-r+1+\frac{2s}{N}j} z_2/z_1)}, \quad (1 \leq j \leq N-1) \quad (\text{A.14})$$

$$E_j(z_1)\Lambda_{j+1}(z_2) = :: \frac{(1 - x^{-r-1-\frac{2s}{N}j} z_2/z_1)}{(1 - x^{r-1-\frac{2s}{N}j} z_2/z_1)}, \quad (1 \leq j \leq N-1) \quad (\text{A.15})$$

$$\Lambda_1(z_1)E_N(z_2) = :: x^{-2r} \frac{(1 - x^{r+1} z_2/z_1)}{(1 - x^{-r+1} z_2/z_1)}, \quad (\text{A.16})$$

$$E_N(z_1)\Lambda_1(z_2) = :: \frac{(1 - x^{-r-1} z_2/z_1)}{(1 - x^{r-1} z_2/z_1)}, \quad (\text{A.17})$$

$$\Lambda_N(z_1)E_N(z_2) = :: x^{2r} \frac{(1 - x^{-r^*-2+2s} z_2/z_1)}{(1 - x^{r^*+2s} z_2/z_1)}, \quad (\text{A.18})$$

$$E_N(z_1)\Lambda_N(z_2) = :: \frac{(1 - x^{r^*+2-2s} z_2/z_1)}{(1 - x^{-r^*-2s} z_2/z_1)}, \quad (\text{A.19})$$

For $N \geq 3$, the normal orderings of $E_j(z)$, $F_j(z)$ are given as followings. The normal orderings for $N = 2$ are summarized in appendix in [13]. For $\text{Re}(r^*) > 0$ we have

$$E_j(z_1)E_j(z_2) = :: z_1^{\frac{2r}{r^*}} (1 - z_2/z_1) \frac{(x^{-2} z_2/z_1; x^{2r^*})_\infty}{(x^{2r^*+2} z_2/z_1; x^{2r^*})_\infty}, \quad (1 \leq j \leq N) \quad (\text{A.20})$$

$$E_j(z_1)E_{j+1}(z_2) = :: (x^{\frac{2s}{N}-j} z_1)^{\frac{r}{r^*}} \frac{(x^{2r-2+\frac{2s}{N}j} z_2/z_1; x^{2r^*})_\infty}{(x^{\frac{2s}{N}-2} z_2/z_1; x^{2r^*})_\infty}, \quad (1 \leq j \leq N)$$

$$E_{j+1}(z_1)E_j(z_2) = :: (x^{\frac{2s}{N}-j-1} z_1)^{\frac{r}{r^*}} \frac{(x^{2r-\frac{2s}{N}j} z_2/z_1; x^{2r^*})_\infty}{(x^{-\frac{2s}{N}} z_2/z_1; x^{2r^*})_\infty}, \quad (1 \leq j \leq N). \quad (\text{A.21})$$

For $\text{Re}(r^*) < 0$ we have

$$E_j(z_1)E_j(z_2) = :: z_1^{\frac{2r^*}{r}} (1 - z_2/z_1) \frac{(x^2 z_2/z_1; x^{-2r^*})_\infty}{(x^{-2r^*-2} z_2/z_1; x^{-2r^*})_\infty}, \quad (1 \leq j \leq N) \quad (\text{A.22})$$

$$\begin{aligned} E_j(z_1)E_{j+1}(z_2) &= :: (x^{\frac{2s}{N}-j} z_1)^{\frac{r^*}{r}} \frac{(x^{-2r^*-2+\frac{2s}{N}} z_2/z_1; x^{-2r^*})_\infty}{(x^{\frac{2s}{N}} z_2/z_1; x^{-2r^*})_\infty}, \quad (1 \leq j \leq N) \\ E_{j+1}(z_1)E_j(z_2) &= :: (x^{\frac{2s}{N}-j-1} z_1)^{\frac{r^*}{r}} \frac{(x^{-2r^*-\frac{2s}{N}} z_2/z_1; x^{-2r^*})_\infty}{(x^{2-\frac{2s}{N}} z_2/z_1; x^{-2r^*})_\infty}, \quad (1 \leq j \leq N). \end{aligned} \quad (\text{A.23})$$

For $\text{Re}(r) > 0$ we have

$$F_j(z_1)F_j(z_2) = :: x^{\frac{2r^*}{r}} (1 - z_2/z_1) \frac{(x^2 z_2/z_1; x^{2r})_\infty}{(x^{2r-2} z_2/z_1; x^{2r})_\infty}, \quad (1 \leq j \leq N) \quad (\text{A.24})$$

$$F_j(z_1)F_{j+1}(z_2) = :: (x^{\frac{2s}{N}-j} z_1)^{-\frac{r^*}{r}} \frac{(x^{2r-2+\frac{2s}{N}} z_2/z_1; x^{2r})_\infty}{(x^{\frac{2s}{N}} z_2/z_1; x^{2r})_\infty}, \quad (1 \leq j \leq N) \quad (\text{A.25})$$

$$F_{j+1}(z_1)F_j(z_2) = :: (x^{\frac{2s}{N}-j-1} z_1)^{-\frac{r^*}{r}} \frac{(x^{2r-2+s} z_2/z_1; x^{2r})_\infty}{(x^{2-\frac{2s}{N}} z_2/z_1; x^{2r})_\infty}, \quad (1 \leq j \leq N). \quad (\text{A.26})$$

For $\text{Re}(r) < 0$ we have

$$F_j(z_1)F_j(z_2) = :: x^{\frac{2r^*}{r}} (1 - z_2/z_1) \frac{(x^{-2} z_2/z_1; x^{-2r})_\infty}{(x^{2-2r} z_2/z_1; x^{-2r})_\infty}, \quad (1 \leq j \leq N) \quad (\text{A.27})$$

$$F_j(z_1)F_{j+1}(z_2) = :: (x^{\frac{2s}{N}-j} z_1)^{-\frac{r^*}{r}} \frac{(x^{-2r+\frac{2s}{N}} z_2/z_1; x^{-2r})_\infty}{(x^{-2+\frac{2s}{N}} z_2/z_1; x^{-2r})_\infty}, \quad (1 \leq j \leq N) \quad (\text{A.28})$$

$$F_{j+1}(z_1)F_j(z_2) = :: (x^{\frac{2s}{N}-j-1} z_1)^{-\frac{r^*}{r}} \frac{(x^{-2r+2-\frac{2s}{N}} z_2/z_1; x^{-2r})_\infty}{(x^{-\frac{2s}{N}} z_2/z_1; x^{-2r})_\infty}, \quad (1 \leq j \leq N). \quad (\text{A.29})$$

For $N \geq 3$ the normal orderings of the screenings $E_j(z)$, $F_j(z)$ are given by

$$E_j(z_1)F_j(z_2) = :: \frac{x^{(1-\frac{2s}{N})2j}}{z_1^2(1 - xz_2/z_1)(1 - x^{-1}z_2/z_1)}, \quad (1 \leq j \leq N), \quad (\text{A.30})$$

$$F_j(z_1)E_j(z_2) = :: \frac{x^{(1-\frac{2s}{N})2j}}{z_1^2(1 - xz_2/z_1)(1 - x^{-1}z_2/z_1)}, \quad (1 \leq j \leq N), \quad (\text{A.31})$$

$$E_j(z_1)F_{j+1}(z_2) = :: x^{(\frac{2s}{N}-1)j} z_1(1 - x^{-1+\frac{2s}{N}} z_2/z_1), \quad (1 \leq j \leq N), \quad (\text{A.32})$$

$$F_{j+1}(z_1)E_j(z_2) = :: x^{(\frac{2s}{N}-1)(j+1)} z_1(1 - x^{1-\frac{2s}{N}} z_2/z_1), \quad (1 \leq j \leq N), \quad (\text{A.33})$$

$$E_{j+1}(z_1)F_j(z_2) = :: x^{(\frac{2s}{N}-1)(j+1)} z_1(1 - x^{-1+\frac{2s}{N}} z_2/z_1), \quad (1 \leq j \leq N), \quad (\text{A.34})$$

$$E_j(z_1)F_{j+1}(z_2) = :: x^{(\frac{2s}{N}-1)j} z_1(1 - x^{1-\frac{2s}{N}} z_2/z_1), \quad (1 \leq j \leq N). \quad (\text{A.35})$$

For $N \geq 3$ we have

$$\mathcal{A}_j(z_1)F_j(z_2) = :: (x^{-j} z_1)^{\frac{2r^*}{r}} \frac{(x^{-r+2+\frac{2s}{N}j} z_2/z_1; x^{2r})_\infty}{(x^{3r-2+\frac{2s}{N}j} z_2/z_1; x^{2r})_\infty},$$

$$(1 \leq j \leq N-1), \quad (\text{A.36})$$

$$F_j(z_1)\mathcal{A}_j(z_2) =:: (x^{(\frac{2s}{N}-1)j} z_1)^{\frac{2r^*}{r}} \frac{(x^{-r+2-\frac{2s}{N}j} z_2/z_1; x^{2r})_\infty}{(x^{3r-2-\frac{2s}{N}j} z_2/z_1; x^{2r})_\infty},$$

$$(1 \leq j \leq N-1), \quad (\text{A.37})$$

$$\mathcal{A}_j(z_1)F_{j+1}(z_2) =:: x^{-r^*(1-\frac{1}{N})}(x^{-j} z_1)^{-\frac{r^*}{r}} \frac{(x^{r-2+\frac{2s}{N}(j+1)} z_2/z_1; x^{2r})_\infty}{(x^{-r+\frac{2s}{N}(j+1)} z_2/z_1; x^{2r})_\infty},$$

$$(1 \leq j \leq N-1), \quad (\text{A.38})$$

$$F_{j+1}(z_1)\mathcal{A}_j(z_2) =:: (x^{(\frac{2s}{N}-1)(j+1)} z_1)^{-\frac{r^*}{r}} \frac{(x^{3r-\frac{2s}{N}(j+1)} z_2/z_1; x^{2r})_\infty}{(x^{r+2-\frac{2s}{N}(j+1)} z_2/z_1; x^{2r})_\infty},$$

$$(1 \leq j \leq N-2), \quad (\text{A.39})$$

$$F_N(z_1)\mathcal{A}_{N-1}(z_2) =:: (x^{2s-N} z_1)^{-\frac{r^*}{r}(1-\frac{1}{N})} \frac{(x^{3r-2s} z_2/z_1; x^{2r})_\infty}{(x^{r+2-2s} z_2/z_1; x^{2r})_\infty}, \quad (\text{A.40})$$

$$\mathcal{A}_j(z_1)F_{j-1}(z_2) =:: x^{r^*(1-\frac{1}{N})}(x^{-j} z_1)^{-\frac{r^*}{r}} \frac{(x^{3r+\frac{2s}{N}(j-1)} z_2/z_1; x^{2r})_\infty}{(x^{r+2+\frac{2s}{N}(j-1)} z_2/z_1; x^{2r})_\infty},$$

$$(1 \leq j \leq N-1), \quad (\text{A.41})$$

$$F_{j-1}(z_1)\mathcal{A}_j(z_2) =:: (x^{(\frac{2s}{N}-1)(j-1)} z_1)^{-\frac{r^*}{r}} \frac{(x^{r-2-\frac{2s}{N}(j-1)} z_2/z_1; x^{2r})_\infty}{(x^{-r-\frac{2s}{N}(j-1)} z_2/z_1; x^{2r})_\infty},$$

$$(2 \leq j \leq N-1), \quad (\text{A.42})$$

$$F_N(z_1)\mathcal{A}_1(z_2) =:: z_1^{-\frac{r^*}{r}(1-\frac{1}{N})} \frac{(x^{r-2} z_2/z_1; x^{2r})_\infty}{(x^{-r} z_2/z_1; x^{2r})_\infty}, \quad (\text{A.43})$$

$$\mathcal{B}_j(z_1)E_j(z_2) =:: (x^{-j} z_1)^{\frac{2r}{r^*}} \frac{(x^{-r^*-2+\frac{2s}{N}j} z_2/z_1; x^{2r^*})_\infty}{(x^{3r^*+2+\frac{2s}{N}j} z_2/z_1; x^{2r^*})_\infty},$$

$$(1 \leq j \leq N-1), \quad (\text{A.44})$$

$$E_j(z_1)\mathcal{B}_j(z_2) =:: (x^{(\frac{2s}{N}-1)j} z_1)^{\frac{2r}{r^*}} \frac{(x^{-r^*-2-\frac{2s}{N}j} z_2/z_1; x^{2r^*})_\infty}{(x^{3r^*+2-\frac{2s}{N}j} z_2/z_1; x^{2r^*})_\infty},$$

$$(1 \leq j \leq N-1), \quad (\text{A.45})$$

$$\mathcal{B}_j(z_1)E_{j+1}(z_2) =:: x^{r(1-\frac{1}{N})}(x^{-j} z_1)^{-\frac{r}{r^*}} \frac{(x^{3r^*+\frac{2s}{N}(j+1)} z_2/z_1; x^{2r^*})_\infty}{(x^{r^*-2+\frac{2s}{N}(j+1)} z_2/z_1; x^{2r^*})_\infty},$$

$$(1 \leq j \leq N-1), \quad (\text{A.46})$$

$$E_{j+1}(z_1)\mathcal{B}_j(z_2) =:: (x^{(\frac{2s}{N}-1)(j+1)} z_1)^{-\frac{r}{r^*}} \frac{(x^{r^*-\frac{2s}{N}(j+1)} z_2/z_1; x^{2r^*})_\infty}{(x^{-r^*+2-\frac{2s}{N}(j+1)} z_2/z_1; x^{2r^*})_\infty},$$

$$(1 \leq j \leq N-2), \quad (\text{A.47})$$

$$E_N(z_1)\mathcal{B}_{N-1}(z_2) =:: (x^{2s-N} z_1)^{-\frac{r}{r^*}(1-\frac{1}{N})} \frac{(x^{r^*-2s} z_2/z_1; x^{2r^*})_\infty}{(x^{-r^*+2-2s} z_2/z_1; x^{2r^*})_\infty}, \quad (\text{A.48})$$

$$\mathcal{B}_j(z_1)E_{j-1}(z_2) =:: x^{-r(1-\frac{1}{N})}(x^{-j}z_1)^{-\frac{r}{r^*}} \frac{(x^{r^*+2+\frac{2s}{N}(j-1)}z_2/z_1; x^{2r^*})_\infty}{(x^{-r^*+\frac{2s}{N}(j-1)}z_2/z_1; x^{2r^*})_\infty},$$

$$(1 \leq j \leq N-1), \quad (\text{A.49})$$

$$E_{j-1}(z_1)\mathcal{B}_j(z_2) =:: (x^{(\frac{2s}{N}-1)(j-1)})^{-\frac{r}{r^*}} \frac{(x^{3r^*-\frac{2s}{N}(j-1)}z_2/z_1; x^{2r^*})_\infty}{(x^{r^*-2-\frac{2s}{N}(j-1)}z_2/z_1; x^{2r^*})_\infty},$$

$$(2 \leq j \leq N-1), \quad (\text{A.50})$$

$$F_N(z_1)\mathcal{A}_1(z_2) =:: z_1^{-\frac{r}{r^*}(1-\frac{1}{N})} \frac{(x^{3r^*}z_2/z_1; x^{2r^*})_\infty}{(x^{r^*-2}z_2/z_1; x^{2r^*})_\infty}. \quad (\text{A.51})$$

For $\text{Re}(r) > 0$ we have

$$F_{j-1}(z_1)H_j(z_2) =:: (x^{(\frac{2s}{N}-1)(j-1)}z_1)^{\frac{1}{r}} \frac{(x^{r-2+\frac{2s}{N}}z_2/z_1; x^{2r})_\infty}{(x^{r+\frac{2s}{N}}z_2/z_1; x^{2r})_\infty}, \quad (\text{A.52})$$

$$F_j(z_1)H_j(z_2) =:: (x^{(\frac{2s}{N}-1)j}z_1)^{-\frac{2}{r}} \frac{(x^{r+2}z_2/z_1; x^{2r})_\infty}{(x^{r-2}z_2/z_1; x^{2r})_\infty}, \quad (\text{A.53})$$

$$H_j(z_1)F_j(z_2) =:: (x^{(\frac{2s}{N}-1)j}z_1)^{-\frac{2}{r}} \frac{(x^{r+2}z_2/z_1; x^{2r})_\infty}{(x^{r-2}z_2/z_1; x^{2r})_\infty}, \quad (\text{A.54})$$

$$H_j(z_1)F_{j+1}(z_2) =:: (x^{(\frac{2s}{N}-1)j}z_1)^{\frac{1}{r}} \frac{(x^{r-2+\frac{2s}{N}}z_2/z_1; x^{2r})_\infty}{(x^{r+\frac{2s}{N}}z_2/z_1; x^{2r})_\infty}, \quad (\text{A.55})$$

For $\text{Re}(r^*) < 0$ we have

$$E_{j-1}(z_1)H_j(z_2) =:: (x^{(\frac{2s}{N}-1)(j-1)}z_1)^{-\frac{1}{r^*}} \frac{(x^{-r^*-2+\frac{2s}{N}}z_2/z_1; x^{-2r^*})_\infty}{(x^{-r^*+\frac{2s}{N}}z_2/z_1; x^{-2r^*})_\infty}, \quad (\text{A.56})$$

$$E_j(z_1)H_j(z_2) =:: (x^{(\frac{2s}{N}-1)j}z_1)^{\frac{2}{r^*}} \frac{(x^{-r^*+2}z_2/z_1; x^{-2r^*})_\infty}{(x^{-r^*-2}z_2/z_1; x^{-2r^*})_\infty}, \quad (\text{A.57})$$

$$H_j(z_1)E_j(z_2) =:: (x^{(\frac{2s}{N}-1)j}z_1)^{\frac{2}{r^*}} \frac{(x^{-r^*+2}z_2/z_1; x^{-2r^*})_\infty}{(x^{-r^*-2}z_2/z_1; x^{-2r^*})_\infty}, \quad (\text{A.58})$$

$$H_j(z_1)E_{j+1}(z_2) =:: (x^{(\frac{2s}{N}-1)j}z_1)^{-\frac{1}{r^*}} \frac{(x^{-r^*-2+\frac{2s}{N}}z_2/z_1; x^{-2r^*})_\infty}{(x^{-r^*+\frac{2s}{N}}z_2/z_1; x^{-2r^*})_\infty}. \quad (\text{A.59})$$

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